Rent seeking, capital accumulation, and macroeconomic growth: Supplementary material

Ben J. Heijdra*
University of Groningen; CESifo

Pim Heijnen†
University of Groningen

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Abstract. This paper contains the supplementary material for Heijdra, B. J. and Heijnen, P. (2025), "Rent-seeking, capital accumulation, and macroeconomic growth," *De Economist*, **173**: ???-???. The MATLAB© programs used to generate the quantitative results are available from the corresponding author upon request.

*Corresponding author. Faculty of Economics and Business, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands. Phone: +31-50-363-7018 (secr), E-mail: b.j.heijdra@rug.nl

†Faculty of Economics and Business, University of Groningen, P. O. Box 800, 9700 AV Groningen, The Netherlands. Phone: +31-50-363-3789, E-mail: p.heijnen@rug.nl

A.1 A dynamic model of rent-seeking

Basic assumptions:

- Diamond-Samuelson overlapping-generations model with (human) capital accumulation and endogenous growth.
- Two generations each of unit size. The population is thus of size 2.
- No bequests so generations are disconnected
- Monopolization after rent seeking activities is certain (no risk).
- The young (superscript y) consume goods $x_{i,t}^{y}$ $(i = 1, 2)$, buy units of the existing capital stock from the old, k_t^y, or a newly produced investment good, z_t^y, from the investment goods sector (both at 'nominal' price Q_t), engage in time-consuming lobbying activities, and enjoy schooling (to augment their human capital stock acquired at birth).
- Timing of decisions during youth:
  - Rent-seeking phase at the start of youth.
  - Consumption and saving decisions.
  - Education phase during the period (human capital installed at the start of the second period).
- The old (superscript o) sell their capital goods to the young, consume goods, $x_{i,t}^{o}$, and work a fraction λ of the time endowment (we set $0 < \lambda < 1$ due to economic ageing and exogenous retirement).

A.1.1 Individual agents

- Continuum of agents indexed by η.
- Lifetime utility function of an agent of type η:

$$\Lambda_t^y(\eta) \equiv \ln c_t^y(\eta) + \beta \ln c_{t+1}^o(\eta), \tag{A.1}$$

where $c_t^y(\eta)$ and $c_{t+1}^y(\eta)$ are defined as:

$$c_t^y(\eta) \equiv \left[\alpha x_{1,t}^y(\eta)^{1-1/\sigma} + (1-\alpha) x_{2,t}^y(\eta)^{1-1/\sigma}\right]^{1/(1-1/\sigma)},$$

$$c_{t+1}^o(\eta) \equiv \left[\alpha x_{1,t+1}^o(\eta)^{1-1/\sigma} + (1-\alpha) x_{2,t+1}^o(\eta)^{1-1/\sigma}\right]^{1/(1-1/\sigma)},$$

where σ is the substitution elasticity between the two goods.

- Budget constraint during youth:

$$P_{1,t}x^y_{1,t}(\eta) + P_{2,t}x^y_{2,t}(\eta) + Q_t\left[z^y_t(\eta) + k^y_t(\eta)\right] = I^y_t(\eta), \tag{A.2}$$

where

$$I^y_t(\eta) = W_t h^y_t(\eta)\left[1 - e_t(\eta) - l_t(\eta)\right] + s_t(\eta)\Pi^m_{1,t}. \tag{A.3}$$

  - W_t is the wage rate on standardized efficiency units of labour.
  - $e_t(\eta)$ is time spent lobbying.
  - $l_t(\eta)$ is time spent on formal schooling.
  - $h^y_t(\eta) = \bar{h}_t$ is the average human capital level in the economy at the start of time t (the young are standing on the shoulders of the old generation).

- Human capital accumulation function:

$$h^o_{t+1}(\eta) = h^y_t(\eta)\left[1 + \phi_e \frac{l_t(\eta)^{1-\theta}}{1-\theta}\right], \qquad \phi_e > 0, \quad 0 < \theta < 1. \tag{A.4}$$

- Budget constraint during old-age:

$$P_{1,t+1}x^o_{1,t+1}(\eta) + P_{2,t+1}x^o_{2,t+1}(\eta) = I^o_{t+1}(\eta), \tag{A.5}$$

with:

$$I^o_{t+1}(\eta) \equiv \lambda W_{t+1}h^o_{t+1}(\eta) + \left[(1-\delta)Q_{t+1} + R^k_{t+1}\right]\left[z^y_t(\eta) + k^y_t(\eta)\right]. \tag{A.6}$$

  - By investing in period t, and owning $z^y_t(\eta) + k^y_t(\eta)$ at the start of old-age, the then old individuals receive a rental income in period $t+1$ equal to R^k_{t+1}. The remaining capital stock they can sell at price Q_{t+1} (to the then young).
  - W_{t+1} is the future wage rate on standardized efficiency units of labour.
  - During old-age only a fraction λ of time is available for working (economic ageing renders people economically useless after a while): $0 < \lambda < 1$

A.1.2 Firms

- Consumption good i is produced with physical and human capital according to:

$$X_{i,t} = \Omega_i H^{\phi_i}_{i,t} K^{1-\phi_i}_{i,t},$$

where Ω_i (> 0) and ϕ_i are constants.

  - Diminishing returns to both factors, i.e. $0 < \phi_i < 1$.

- We can have the same technology for goods 1 and 2 (but we state the most general model here).
- Both factors are perfectly mobile across sectors.

- The total and marginal cost functions are:

$$\begin{aligned} TC_i^x(W_t, R_t^k, X_{i,t}) &\equiv MC_i^x(W_t, R_t^k) X_{i,t}, \\ MC_i^x(W_t, R_t^k) &\equiv \left(\frac{W_t}{\phi_i}\right)^{\phi_i} \left(\frac{R_t^k}{1-\phi_i}\right)^{1-\phi_i} \frac{1}{\Omega_i}, \\ &= P_{2,t} \left(\frac{w_t}{\phi_i}\right)^{\phi_i} \left(\frac{r_t^k}{1-\phi_i}\right)^{1-\phi_i} \frac{1}{\Omega_i} \\ &\equiv P_{2,t} mc_i^x(w_t, r_t^k), \end{aligned}$$

with:

$$r_t^k \equiv \frac{R_t^k}{P_{2,t}}, \qquad w_t \equiv \frac{W_t}{P_{2,t}}. \tag{A.7}$$

- The derived demands for units of physical and human capital are obtained by employing Shephard's Lemma:

$$\begin{aligned} H_{i,t} &= \frac{\partial MC_i^x(W_t, R_t^k)}{\partial W_t} X_{i,t} = \frac{\phi_i MC_i^x(W_t, R_t^k)}{W_t} X_{i,t}, \\ K_{i,t} &= \frac{\partial MC_i^x(W_t, R_t^k)}{\partial R_t^k} X_{i,t} = \frac{(1-\phi_i) MC_i^x(W_t, R_t^k)}{R_t^k} X_{i,t}. \end{aligned}$$

- By substituting the production function and rearranging the resulting expression we find:

$$\begin{aligned} W_t &= \phi_i MC_i^x(W_t, R_t^k) \Omega_i \kappa_{i,t}^{1-\phi_i}, \\ R_t^k &= (1-\phi_i) MC_i^x(W_t, R_t^k) \Omega_i \kappa_{i,t}^{-\phi_i}, \end{aligned}$$

with:

$$\kappa_{i,t} \equiv \frac{K_{i,t}}{H_{i,t}}. \tag{A.8}$$

- Profit in sector i is:

$$\Pi_{i,t} = P_{i,t} X_{i,t} - MC_i^x(W_t, R_t^k) X_{i,t}.$$

- For good X_2 (which is always produced competitively) we find that $P_{2,t} = MC_2^x(W_t, R_t^k)$. By using X_2 as the numeraire commodity we find that:

$$P_{2,t} = P_{2,t} mc_2^x(w_t, r_t^k) \qquad \Leftrightarrow \qquad mc_2^x(w_t, r_t^k) = 1.$$

- In summary, factor demands can be written as:

$$w_t = \phi_1 mc_1^x(w_t, r_t^k)\Omega_1 \kappa_{1,t}^{1-\phi_1} = \phi_2 \Omega_2 \kappa_{2,t}^{1-\phi_2},$$
$$r_t^k = (1-\phi_1) mc_1^x(w_t, r_t^k)\Omega_1 \kappa_{1,t}^{-\phi_1} = (1-\phi_2)\Omega_2 \kappa_{2,t}^{-\phi_2}.$$

- Since good x_2 is produced competively, $P_{2,t} = MC_2^x(W_t, R_t^k)$ and we find (by eliminating $\kappa_{2,t}$ from the factor demands) that:

$$1 = \left(\frac{w_t}{\phi_2}\right)^{\phi_2} \left(\frac{r_t^k}{1-\phi_2}\right)^{1-\phi_2} \frac{1}{\Omega_2}.$$

- For good x_1 we find (by eliminating $\kappa_{1,t}$ from the factor demands) that real marginal cost equals:

$$mc_1^x(w_t, r_t^k) = \left(\frac{w_t}{\phi_1}\right)^{\phi_1} \left(\frac{r_t^k}{1-\phi_1}\right)^{1-\phi_1} \frac{1}{\Omega_1}.$$

  Hence, if x_1 is also produced competitively we find that $p_t \equiv P_{1,t}/P_{2,t} = mc_1^x(w_t, r_t^k)$.

- The total stock of efficiency units of labour is:

$$H_t \equiv \int_{\eta_L}^{\eta_H} \left[\lambda h_t^o(\eta) + \left[1 - e_t(\eta) - l_t(\eta)\right] h_t^y(\eta)\right] dF(\eta).$$

  - Units of ‘old’ and ‘young’ human capital are perfect substitutes.

- The investment good is also produced with units of physical and human capital:

$$Z_t = \Omega_z H_{z,t}^{\psi} K_{z,t}^{1-\psi}.$$

- The representative firm hires these inputs (from their owners) to maximize profit:

$$\Pi_t^z \equiv Q_t Z_t - W_t H_{z,t} - R_t^k K_{z,t},$$

  which gives:

$$R_t^k = (1-\psi) Q_t \Omega_z H_{z,t}^{\psi} K_{z,t}^{-\psi},$$
$$W_t = \psi Q_t \Omega_z H_{z,t}^{\psi-1} K_{z,t}^{1-\psi}.$$

- Again using consumption good x_2 as the numeraire commodity we find:

$$r_t^k = (1-\psi) q_t \Omega_z \kappa_{z,t}^{-\psi},$$
$$w_t = \psi q_t \Omega_z \kappa_{z,t}^{1-\psi},$$

with:

$$q_t \equiv \frac{Q_t}{P_{2,t}}. \tag{A.9}$$

- Total cost and marginal cost functions are:

$$TC^z(W_t, R_t^k, Z_t) \equiv \left(\frac{W_t}{\psi}\right)^{\psi} \left(\frac{R_t^k}{1-\psi}\right)^{1-\psi} \frac{Z_t}{\Omega_z},$$
$$MC^z(W_t, R_t^k, Z_t) \equiv \left(\frac{W_t}{\psi}\right)^{\psi} \left(\frac{R_t^k}{1-\psi}\right)^{1-\psi} \frac{1}{\Omega_z} = P_{2,t} \left(\frac{w_t}{\psi}\right)^{\psi} \left(\frac{r_t^k}{1-\psi}\right)^{1-\psi} \frac{1}{\Omega_z}$$

- Since the investment good is produced competitively we have that:

$$Q_t = MC^z(W_t, R_t^k) \qquad \Leftrightarrow \qquad q_t = mc^z(w_t, r_t^k) \equiv \left(\frac{w_t}{\psi}\right)^{\psi} \left(\frac{r_t^k}{1-\psi}\right)^{1-\psi} \frac{1}{\Omega_z}.$$

A.1.3 Loose ends

- Physical capital accumulation:

$$K_{t+1} = Z_t + (1-\delta)K_t. \tag{A.10}$$

- Stock of human capital available for productive use:

$$H_t = \bar{h}_t \left[1 + \lambda - \bar{e}_t - \bar{l}_t\right], \tag{A.11}$$

with:

$$\bar{e}_t \equiv \int_{\eta_L}^{\eta_H} e_t(\eta) dF(\eta), \qquad \bar{l}_t \equiv \int_{\eta_L}^{\eta_H} l_t(\eta) dF(\eta). \tag{A.12}$$

- Equilibrium in the investment goods market:

$$Z_t = \int_{\eta_L}^{\eta_H} z_t^y(\eta) dF(\eta). \tag{A.13}$$

- Equilibrium in the market for used capital goods:

$$\int_{\eta_L}^{\eta_H} k_t^y(\eta) dF(\eta) = (1-\delta)K_t. \tag{A.14}$$

- Equilibrium condition in the physical capital rental market:

$$K_t = K_{1,t} + K_{2,t} + K_{z,t}.$$

- Equilibrium condition in the human capital rental market:

$$H_t = H_{1,t} + H_{2,t} + H_{z,t}.$$

A.1.4 Model solution

- We employ two-stage budgeting to solve the individual's utility maximization problem.
- Given the structure of preferences, we know that:

$$X_t^y(\eta) \equiv P_{1,t}x_{1,t}^y(\eta) + P_{2,t}x_{2,t}^y(\eta) = P_{V,t}c_t^y(\eta),$$
$$X_{t+1}^o(\eta) \equiv P_{1,t+1}x_{1,t+1}^o(\eta) + P_{2,t+1}x_{2,t+1}^o(\eta) = P_{V,t+1}c_{t+1}^o(\eta),$$

where $X_t^y(\eta)$ is 'full' consumption and $P_{V,t}$ is the true price index:

$$P_{V,t} \equiv \left[\alpha^\sigma P_{1,t}^{1-\sigma} + (1-\alpha)^\sigma P_{2,t}^{1-\sigma}\right]^{1/(1-\sigma)}.$$

- Useful results from duality theory:
  - The expenditure functions are $E_t^y(\eta) \equiv P_{V,t}c_t^y(\eta)$ and $E_{t+1}^o(\eta) \equiv P_{V,t+1}c_{t+1}^o(\eta)$ so we can recover the *Hicksian* demands for the underlying goods in the usual fashion (Shephard's Lemma):

$$x_{i,t}^y(\eta) = \frac{\partial E_t^y(\eta)}{\partial P_{i,t}} = \frac{\partial P_{V,t}}{\partial P_{i,t}}c_t^y(\eta), \qquad x_{i,t+1}^o(\eta) = \frac{\partial E_{t+1}^o(\eta)}{\partial P_{i,t+1}} = \frac{\partial P_{V,t+1}}{\partial P_{i,t+1}}c_{t+1}^o(\eta).$$

  - The indirect (sub)utility functions are $V_t^y(\eta) \equiv X_t^y(\eta)/P_{V,t}$ and $V_{t+1}^o(\eta) \equiv X_{t+1}^o(\eta)/P_{V,t+1}$ and the Marshallian demands for the underlying goods in the usual fashion (Roy's Identity):

$$x_{i,t}^y(\eta) = -\frac{\partial V_t^y(\eta)/\partial P_{i,t}}{\partial V_t^y(\eta)/\partial X_t^y(\eta)} = \frac{\partial P_{V,t}}{\partial P_{i,t}}\frac{X_t^y(\eta)}{P_{V,t}}, \qquad x_{i,t+1}^o(\eta) = -\frac{\partial V_{t+1}^o(\eta)/\partial P_{i,t+1}}{\partial V_{t+1}^o(\eta)/\partial X_{t+1}^o(\eta)} = \frac{\partial P_{V,t+1}}{\partial P_{i,t+1}}\frac{X_{t+1}^o(\eta)}{P_{V,t+1}}.$$

- Budget constraints for young and old:

$$P_{V,t}c_t^y(\eta) + Q_t\left[z_t^y(\eta) + k_t^y(\eta)\right] = W_t h_t^y(\eta)\left[1 - e_t(\eta) - l_t(\eta)\right] + s_t(\eta)\Pi_{1,t}^m,$$
$$P_{V,t+1}c_{t+1}^o(\eta) = \lambda W_{t+1}h_{t+1}^o(\eta) + \left[(1-\delta)Q_{t+1} + R_{t+1}^k\right]\left[z_t^y(\eta) + k_t^y(\eta)\right].$$

- Define the 'nominal' interest rate as:

$$1 + R_{t+1}^n \equiv \frac{(1-\delta)Q_{t+1} + R_{t+1}^k}{Q_t}. \tag{A.15}$$

- Solve the old-age budget constraint for $[z_t^y(\eta)+k_t^y(\eta)]$:

$$Q_t\left[z_t^y(\eta)+k_t^y(\eta)\right]=\frac{P_{V,t+1}c_{t+1}^o(\eta)-\lambda W_{t+1}h_{t+1}^o(\eta)}{1+R_{t+1}^n}.$$

- Substitute into the youth budget constraint to get the consolidated budget constraint in nominal terms:

$$P_{V,t}c_t^y(\eta)+\frac{P_{V,t+1}c_{t+1}^o(\eta)}{1+R_{t+1}^n}=HW_t^y(\eta), \tag{A.16}$$

where human wealth during youth is:

$$HW_t^y(\eta)\equiv W_th_t^y(\eta)\left[1-e_t(\eta)-l_t(\eta)\right]+\frac{\lambda W_{t+1}h_{t+1}^o(\eta)}{1+R_{t+1}^n}+s_t(\eta)\Pi_{1,t}^m.$$

- The remaining constraint:

$$h_{t+1}^o(\eta)=\bar{h}_t\left[1+\phi_e\frac{l_t(\eta)^{1-\theta}}{1-\theta}\right]. \tag{A.17}$$

where we have used the initial condition $h_t^y(\eta)=\bar{h}_t$.

- Since there is no uncertainty (monopolization is for sure) we can solve the optimization problem in one go. In particular, the agents chooses $c_t^y(\eta)$, $c_{t+1}^o(\eta)$, $l_t(\eta)$, and $e_t(\eta)$ [and thus also $h_{t+1}^o(\eta)$] to maximize (A.1) subject to the budget constraint (A.16) and the human capital accumulation identity (A.17).

- Note: we continue to use 'nominal' terms (and use the numeraire, $P_{2,t}=1$, right at the end).

- The Lagrangian is:

$$\begin{aligned}\mathcal{L}_t^y\equiv \ln c_t^y(\eta)+\beta\ln c_{t+1}^o(\eta)+\mu_t\Bigg[&s_t(\eta)\Pi_{1,t}^m+W_t\bar{h}_t\left[1-e_t(\eta)-l_t(\eta)\right]\\&+\frac{\lambda W_{t+1}}{1+R_{t+1}^n}\bar{h}_t\left(1+\phi_e\frac{l_t(\eta)^{1-\theta}}{1-\theta}\right)-P_{V,t}c_t^y(\eta)-\frac{P_{V,t+1}c_{t+1}^o(\eta)}{1+R_{t+1}^n}\Bigg].\end{aligned}$$

- First-order conditions:

$$\begin{aligned}\frac{\partial\mathcal{L}_t^y}{\partial c_t^y(\eta)}&=\frac{1}{c_t^y(\eta)}-\mu_tP_{V,t}=0,\\ \frac{\partial\mathcal{L}_t^y}{\partial c_{t+1}^o(\eta)}&=\frac{\beta}{c_{t+1}^o(\eta)}-\frac{\mu_tP_{V,t+1}}{1+R_{t+1}^n}=0,\\ \frac{\partial\mathcal{L}_t^y}{\partial l_t(\eta)}&=\mu_t\left[-1+\frac{W_{t+1}}{(1+R_{t+1}^n)W_t}\lambda\phi_el_t(\eta)^{-\theta}\right]W_t\bar{h}_t=0,\end{aligned}$$

$$\frac{\partial \mathcal{L}_t^y}{\partial e_t(\eta)} = \mu_t \left[\Pi_{1,t}^m \frac{\partial s_t(\eta)}{\partial e_t(\eta)} - W_t \bar{h}_t \right] = 0.$$

- Substituting the first two into the budget constraint (A.16) we find:

$$P_{V,t} c_t^y(\eta) = \frac{1}{1+\beta} HW_t^y(\eta),$$
$$\frac{P_{V,t+1} c_{t+1}^o(\eta)}{1+R_{t+1}^n} = \frac{\beta}{1+\beta} HW_t^y(\eta).$$

- Clearly, it follows from the third first-order condition that every agent chooses the same amount of schooling:

$$l_t(\eta) = \bar{l}_t = l_t \equiv \left[\frac{\lambda \phi_e W_{t+1}}{(1+R_{t+1}^n) W_t} \right]^{1/\theta}. \tag{A.18}$$

- For the success function, $s_t(\eta) = \eta e_t(\eta)^{\varepsilon} / E_t$, we find from the fourth first-order condition that:

$$\varepsilon \Pi_{1,t}^m \frac{\eta e_t(\eta)^{\varepsilon-1}}{E_t} = W_t \bar{h}_t \qquad \Leftrightarrow \qquad e_t(\eta) = \left[\frac{\varepsilon \Pi_{1,t}^m}{W_t \bar{h}_t} \frac{\eta}{E_t} \right]^{1/(1-\varepsilon)}. \tag{A.19}$$

- It follows that total rent-seeking effort E_t and wasted labour $\bar{e}_t$ amount to:

$$E_t \equiv \int_{\eta_L}^{\eta_H} \eta e_t(\eta)^{\varepsilon} dF(\eta) = \left[\frac{\varepsilon \Pi_{1,t}^m}{W_t \bar{h}_t E_t} \right]^{\varepsilon/(1-\varepsilon)} \int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta),$$
$$\bar{e}_t \equiv \int_{\eta_L}^{\eta_H} e_t(\eta) dF(\eta) = \left[\frac{\varepsilon \Pi_{1,t}^m}{W_t \bar{h}_t E_t} \right]^{1/(1-\varepsilon)} \int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta).$$

- Solving the first of these expressions for E_t gives:

$$E_t = \left[\frac{\varepsilon \Pi_{1,t}^m}{W_t \bar{h}_t} \right]^{\varepsilon} \left[\int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta) \right]^{1-\varepsilon}. \tag{A.20}$$

- Using (A.20) in (A.19) we find that $e_t(\eta)$ and $s_t(\eta)$ can be written as:

$$e_t(\eta) = s_t(\eta) \bar{e}_t, \qquad s_t(\eta) \equiv \frac{\eta^{1/(1-\varepsilon)}}{\int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta)}, \tag{A.21}$$

where $\bar{e}_t$ is given by:

$$\bar{e}_t = \frac{\varepsilon \Pi_{1,t}^m}{W_t \bar{h}_t}. \tag{A.22}$$

- Optimal choices can be written as follows:

$$
\begin{aligned}
P_{V,t}c_t^y(\eta) &= \frac{1}{1+\beta}HW_t^y(\eta),\\
\frac{P_{V,t+1}c_{t+1}^o(\eta)}{1+R_{t+1}^n} &= \frac{\beta}{1+\beta}HW_t^y(\eta),\\
Q_t\left[z_t^y(\eta)+k_t^y(\eta)\right] &= s_t(\eta)\Pi_{1,t}^m + W_t\bar{h}_t\left[1-e_t(\eta)-l_t\right] - \frac{1}{1+\beta}HW_t^y(\eta),\\
&= \frac{\beta}{1+\beta}\left[s_t(\eta)\Pi_{1,t}^m + W_t\bar{h}_t\left[1-e_t(\eta)-l_t\right]\right] - \frac{1}{1+\beta}\frac{\lambda W_{t+1}}{1+R_{t+1}^n}\bar{h}_t\left[1+\phi_e\frac{l_t^{1-\theta}}{1-\theta}\right],\\
HW_t^y(\eta) &\equiv s_t(\eta)\Pi_{1,t}^m + W_t\bar{h}_t\left[1-e_t(\eta)-l_t\right] + \frac{\lambda W_{t+1}}{1+R_{t+1}^n}\bar{h}_t\left[1+\phi_e\frac{l_t^{1-\theta}}{1-\theta}\right],\\
l_t &= \left[\frac{\lambda\phi_e W_{t+1}}{(1+R_{t+1}^n)W_t}\right]^{1/\theta}.
\end{aligned}
$$

- Note that:

$$
\int_{\eta_L}^{\eta_H}\left[z_t^y(\eta)+k_t^y(\eta)\right]dF(\eta) = Z_t + (1-\delta)K_t = K_{t+1},
$$

where we have used (A.10), (A.13), and (A.14).

- Using (A.22) aggregate saving can be rewritten as:

$$
\begin{aligned}
Q_tK_{t+1} = &\frac{\beta}{1+\beta}\left[(1-\varepsilon)\Pi_{1,t}^m + W_t\bar{h}_t\left[1-l_t\right]\right]\\
&-\frac{1}{1+\beta}\frac{\lambda W_{t+1}}{1+R_{t+1}^n}\bar{h}_t\left[1+\phi_e\frac{l_t^{1-\theta}}{1-\theta}\right].
\end{aligned}
\tag{A.23}
$$

- It follows (by using (A.10)) that the demand for new capital goods is:

$$
\begin{aligned}
Q_tZ_t = &\frac{\beta}{1+\beta}\left[(1-\varepsilon)\Pi_{1,t}^m + W_t\bar{h}_t\left[1-l_t\right]\right]\\
&-\frac{1}{1+\beta}\frac{\lambda W_{t+1}}{1+R_{t+1}^n}\bar{h}_t\left[1+\phi_e\frac{l_t^{1-\theta}}{1-\theta}\right] - Q_t(1-\delta)K_t
\end{aligned}
$$

- Aggregate demands for composite consumption goods:

$$
\begin{aligned}
P_{V,t}c_t^y &= \frac{1}{1+\beta}HW_t^y\\
\frac{P_{V,t+1}c_{t+1}^o}{1+R_{t+1}^n} &= \frac{\beta}{1+\beta}HW_t^y
\end{aligned}
$$

- Aggregate human wealth of the young (after using (A.22)):

$$HW_t^y \equiv (1-\varepsilon)\Pi_{1,t}^m + W_t \bar{h}_t \left[1-l_t\right] + \frac{\lambda W_{t+1}}{1+R_{t+1}^n}\bar{h}_t \left[1+\phi_e \frac{l_t^{1-\theta}}{1-\theta}\right].$$

- Demand in sector 1 originates from the young and the old.

  – The young cohort's demand for good 1:

$$X_{1,t}^y = \frac{\partial P_{V,t}}{\partial P_{1,t}} c_t^y = \frac{\partial P_{V,t}}{\partial P_{1,t}} \frac{1}{1+\beta}\frac{HW_t^y}{P_{V,t}} = \frac{\alpha^\sigma P_{1,t}^{-\sigma}}{\alpha^\sigma P_{1,t}^{1-\sigma} + (1-\alpha)^\sigma P_{2,t}^{1-\sigma}} \frac{HW_t^y}{1+\beta}.$$

  By holding HW_t^y constant this is interpreted as a Marshallian demand curve.

  – The old cohort's demand for good 1:

$$X_{1,t}^o = \frac{\alpha^\sigma P_{1,t}^{-\sigma}}{\alpha^\sigma P_{1,t}^{1-\sigma} + (1-\alpha)^\sigma P_{2,t}^{1-\sigma}} I_t^o,$$
$$I_t^o = \lambda W_t h_t^o + \left[(1-\delta)\,Q_t + R_t^k\right] K_t.$$

  – Total demand is thus:

$$X_{1,t} = \frac{\alpha^\sigma P_{1,t}^{-\sigma}}{\alpha^\sigma P_{1,t}^{1-\sigma} + (1-\alpha)^\sigma P_{2,t}^{1-\sigma}} \left[\frac{HW_t^y}{1+\beta} + I_t^o\right].$$

- The monopolist in sector 1 has the following profit function:

$$\Pi_{1,t}^m = \left[P_{1,t} - MC_1^x(W_t, R_t^k)\right] X_{1,t}$$

and the monopoly price is set according to the usual markup rule:

$$P_{1,t}^m = \mu_{1,t}^m MC_1^x(W_t, R_t^k), \qquad \mu_{1,t}^m \equiv \frac{\varepsilon_{d,t}^m}{\varepsilon_{d,t}^m - 1} > 1,$$
$$\varepsilon_{d,t}^m \equiv -\frac{\partial X_{1,t}}{\partial P_{1,t}}\frac{P_{1,t}}{X_{1,t}} = \frac{\alpha^\sigma (P_{1,t}^m)^{1-\sigma} + \sigma(1-\alpha)^\sigma (P_{2,t}^c)^{1-\sigma}}{\alpha^\sigma (P_{1,t}^m)^{1-\sigma} + (1-\alpha)^\sigma (P_{2,t}^c)^{1-\sigma}},$$
$$= \frac{\alpha^\sigma (p_t^m)^{1-\sigma} + \sigma(1-\alpha)^\sigma}{\alpha^\sigma (p_t^m)^{1-\sigma} + (1-\alpha)^\sigma} > 1,$$
$$\mu_{1,t}^m = \frac{\alpha^\sigma (p_t^m)^{1-\sigma} + \sigma(1-\alpha)^\sigma}{(\sigma-1)\,(1-\alpha)^\sigma},$$

with:

$$p_t^m \equiv \frac{P_{1,t}^m}{P_{2,t}^c}. \qquad \text{(A.24)}$$

- For future use we note that:

$$\mu_{1,t}^m - 1 = \frac{\alpha^\sigma (p_t^m)^{1-\sigma} + (1-\alpha)^\sigma}{(\sigma - 1)(1-\alpha)^\sigma}.$$

- Using the expression for $MC_1^x(W_t, R_t^k)$ derived above we find:

$$p_t^m = \mu_{1,t}^m mc_1^x(w_t, r_t^k),$$

where $mc_1^x(w_t, r_t^k)$ is real marginal cost in the monopolistic sector (see above):

$$mc_1^x(w_t, r_t^k) \equiv \left(\frac{w_t}{\phi_1}\right)^{\phi_1} \left(\frac{r_t^k}{1-\phi_1}\right)^{1-\phi_1} \frac{1}{\Omega_1}.$$

- It follows that $\varepsilon_{d,t}^m$ can be written as:

$$\varepsilon_{d,t}^m = \frac{\alpha^\sigma \left(\frac{\varepsilon_{d,t}^m}{\varepsilon_{d,t}^m - 1} mc_1^x(w_t, r_t^k)\right)^{1-\sigma} + \sigma(1-\alpha)^\sigma}{\alpha^\sigma \left(\frac{\varepsilon_{d,t}^m}{\varepsilon_{d,t}^m - 1} mc_1^x(w_t, r_t^k)\right)^{1-\sigma} + (1-\alpha)^\sigma}.$$

- In real terms the factor demand are:

$$r_t^k = (1-\phi_1) mc_1^x(w_t, r_t^k)\Omega_1 \kappa_{1,t}^{-\phi_1},$$
$$w_t = \phi_1 mc_1^x(w_t, r_t^k)\Omega_1 \kappa_{1,t}^{1-\phi_1}.$$

- Aggregate profit equals:

$$\begin{aligned}\Pi_{1,t}^m &= X_{1,t}\left(P_{1,t} - MC_1^x(W_t, R_t^k)\right), \\ &= \left(\mu_{1,t}^m - 1\right) MC_1^x(W_t, R_t^k) X_{1,t}, \\ &= \Xi_t \left[\frac{HW_t^y}{1+\beta} + I_t^o\right], \end{aligned} \tag{A.25}$$

where Ξ_t is an auxiliary term:

$$\Xi_t \equiv \frac{\alpha^\sigma (p_t^m)^{1-\sigma}}{\alpha^\sigma (p_t^m)^{1-\sigma} + \sigma(1-\alpha)^\sigma}. \tag{A.26}$$

- Properties:
  - in the competitive case, $P_{1,t} = MC_1^x(W_t, R_t^k)$ (so that $\mu_{1,t}^m = 1$) and $\Xi_t = 0$ for all t.
  - in the monopoly case, $0 < \Xi_t < 1$.
  - since $\mu_{1,t}^m$ depends on p_t^m so does Ξ_t!

- We find (after using (A.22)) that:

$$\frac{HW_t^y}{1+\beta} + I_t^o = \frac{1}{1+\beta}\left((1-\varepsilon)\Pi_{1,t}^m + W_t\bar{h}_t\left[1-l_t\right] + \frac{\lambda W_{t+1}}{1+R_{t+1}^n}\bar{h}_t\left[1+\phi_e\frac{l_t^{1-\theta}}{1-\theta}\right]\right)$$
$$+\lambda W_t\bar{h}_t + \left[(1-\delta)\,Q_t + R_t^k\right]K_t. \tag{A.27}$$

Aggregate profit positively affects total wealth and vice versa, so current profit depends in part on itself because young agents consume part of it.

- By solving (A.25) and (A.27) for $\Pi_{1,t}^m$ and $\frac{HW_t^y}{1+\beta} + I_t^o$ we find:

$$\Pi_{1,t}^m = \frac{\Xi_t}{1+\beta-(1-\varepsilon)\Xi_t}\left[W_t\bar{h}_t\left[1-l_t\right] + \frac{\lambda W_{t+1}}{1+R_{t+1}^n}\bar{h}_t\left(1+\phi_e\frac{l_t^{1-\theta}}{1-\theta}\right)\right.$$
$$\left.+(1+\beta)\left[\lambda W_t\bar{h}_t + \left[(1-\delta)\,Q_t + R_t^k\right]K_t\right]\right], \tag{A.28}$$

and:

$$\frac{HW_t^y}{1+\beta} + I_t^o = \frac{1}{1+\beta-(1-\varepsilon)\Xi_t}\left[W_t\bar{h}_t\left[1-l_t\right] + \frac{\lambda W_{t+1}}{1+R_{t+1}^n}\bar{h}_t\left(1+\phi_e\frac{l_t^{1-\theta}}{1-\theta}\right)\right.$$
$$\left.+(1+\beta)\left[\lambda W_t\bar{h}_t + \left[(1-\delta)\,Q_t + R_t^k\right]K_t\right]\right]. \tag{A.29}$$

- It follows that $\Pi_{1,t}^m$ and $\frac{HW_t^y}{1+\beta} + I_t^o$ are both proportional to the growing variables $\bar{h}_t$ and K_t.

- Demand for good 2 originates from the young and the old.

  - The young cohort's demand for good 2:

$$X_{2,t}^y = \frac{\partial P_{V,t}}{\partial P_{2,t}}c_t^y = \frac{\partial P_{V,t}}{\partial P_{2,t}}\frac{1}{1+\beta}\frac{HW_t^y}{P_{V,t}} = \frac{(1-\alpha)^\sigma P_{2,t}^{-\sigma}}{\alpha^\sigma P_{1,t}^{1-\sigma} + (1-\alpha)^\sigma P_{2,t}^{1-\sigma}}\frac{HW_t^y}{1+\beta}.$$

  - The old cohort's demand for good 2:

$$X_{2,t}^o = \frac{(1-\alpha)^\sigma P_{2,t}^{-\sigma}}{\alpha^\sigma P_{1,t}^{1-\sigma} + (1-\alpha)^\sigma P_{2,t}^{1-\sigma}}I_t^o,$$
$$I_t^o = \lambda W_t\bar{h}_t + \left[(1-\delta)\,Q_t + R_t^k\right]K_t.$$

  - Total demand for good 2 is thus:

$$X_{2,t} = \frac{(1-\alpha)^\sigma P_{2,t}^{-\sigma}}{\alpha^\sigma P_{1,t}^{1-\sigma} + (1-\alpha)^\sigma P_{2,t}^{1-\sigma}}\left[\frac{HW_t^y}{1+\beta} + I_t^o\right].$$

A.1.4.1 Verify Walras Law

- Spending at time t:

$$P_{V,t}\left[c_t^y + c_t^o\right] + Q_t\left[Z_t + (1-\delta)K_t\right] = W_t\bar{h}_t\left[1 - \bar{e}_t - l_t\right] + \Pi_{1,t}^m + \lambda W_t h_t^o + \left[(1-\delta)Q_t + R_t^k\right]K_t.$$

- Simplifying:

$$P_{V,t}\left[c_t^y + c_t^o\right] + Q_t Z_t = W_t\bar{h}_t\left[1 - \bar{e}_t - l_t\right] + \Pi_{1,t}^m + \lambda W_t h_t^o + R_t^k K_t.$$

- But $P_{V,t}\left[c_t^y + c_t^o\right] = P_{1,t}X_{1,t} + P_{2,t}X_{2,t}$ and $H_t = \bar{h}_t\left[1 - \bar{e}_t - l_t\right] + \lambda h_t^o$ so we get:

$$P_{1,t}X_{1,t} + P_{2,t}X_{2,t} + Q_t Z_t = W_t H_t + \Pi_{1,t}^m + R_t^k K_t.$$

- But $\Pi_{1,t}^m = (P_{1,t} - MC_1^x(W_t, R_t^k))X_{1,t}$, $P_{2,t} = MC_2^x(W_t, R_t^k)$, and $Q_t = MC^z(W_t, R_t^k)$ so we get:

$$MC_1^x(W_t, R_t^k)X_{1,t} + MC_2^x(W_t, R_t^k)X_{2,t} + MC^z(W_t, R_t^k)Z_t = W_t H_t + R_t^k K_t$$

  Right-hand side: total factor income. Left-hand side: total spending on consumption and investment goods evaluated at the true marginal cost of producing these goods.

A.1.4.2 Checking market equilibrium conditions

In the numerical model (for debugging purposes) we conduct some consistency checks by computing the same quantity in two different ways.

- Market for good 1 (demand and supply):

$$\begin{aligned}\frac{X_{1,t}}{\bar{h}_t} &= \frac{\alpha^\sigma p_{1,t}^{-\sigma}}{\alpha^\sigma p_{1,t}^{1-\sigma} + (1-\alpha)^\sigma}\left[\frac{1}{1+\beta}\left((1-\varepsilon)\frac{\pi_{1,t}^m}{\bar{h}_t} + w_t\left[1 - l_t\right] + \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1+r_{t+1}}\right)\right. \\ &\quad \left. + \lambda w_t + \left[(1-\delta)\,q_t + r_t^k\right]\frac{K_t}{\bar{h}_t}\right], \\ \frac{X_{1,t}}{H_t} &= \Omega_1 u_{1,t}\kappa_{1,t}^{1-\phi_1}.\end{aligned}$$

- Market for good 2 (demand and supply):

$$\begin{aligned}\frac{X_{2,t}}{\bar{h}_t} &= \frac{(1-\alpha)^\sigma}{\alpha^\sigma p_{1,t}^{1-\sigma} + (1-\alpha)^\sigma}\left[\frac{1}{1+\beta}\left((1-\varepsilon)\frac{\pi_{1,t}^m}{\bar{h}_t} + w_t\left[1 - l_t\right] + \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1+r_{t+1}}\right)\right. \\ &\quad \left. + \lambda w_t + \left[(1-\delta)\,q_t + r_t^k\right]\frac{K_t}{\bar{h}_t}\right],\end{aligned}$$

$$\frac{X_{2,t}}{H_t} = \Omega_2 u_{2,t} \kappa_{2,t}^{1-\phi_2}.$$

- Market for investment goods (demand and supply):

$$q_t \frac{Z_t}{\bar{h}_t} = \frac{\beta}{1+\beta}\left[(1-\varepsilon)\frac{\pi_{1,t}^m}{\bar{h}_t} + w_t\left[1-l_t\right]\right] - \frac{1}{1+\beta}\frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1+r_{t+1}} - q_t(1-\delta)\frac{K_t}{\bar{h}_t},$$

$$\frac{Z_t}{H_t} = u_{z,t}\Omega_z \kappa_{z,t}^{1-\psi}.$$

- Aggregate output:

$$\frac{Y_t}{H_t} = p_t \frac{X_{1,t}}{H_t} + \frac{X_{2,t}}{H_t} + q_t \frac{Z_t}{H_t}.$$

A.1.5 The dynamic rent-seeking equilibrium

- Summary of the model: see Table A.2
- There are 23 endogenous variables and 23 equations so all should be swell.
- Insight #1: if x_1 and x_2 are identical from the production side then the real monopoly price is constant (with or without rent seeking)!
  - Suppose that $\phi_1 = \phi_2 = \phi$ and $\Omega_1 = \Omega_2 = \Omega$.
  - Then (TA4.7) and (TA4.10) together imply that:

$$mc_{1,t}^x = \left(\frac{w_t}{\phi_1}\right)^{\phi_1}\left(\frac{r_t^k}{1-\phi_1}\right)^{1-\phi_1}\frac{1}{\Omega_1}.$$

  - But (TA4.8) and (TA4.11) together imply that:

$$1 = \left(\frac{w_t}{\phi_2}\right)^{\phi_2}\left(\frac{r_t^k}{1-\phi_2}\right)^{1-\phi_2}\frac{1}{\Omega_2}.$$

  - Hence, since $\phi_1 = \phi_2$ and $\Omega_1 = \Omega_2$ we find that:

$$mc_{1,t}^x = 1, \quad \kappa_{1,t} = \kappa_{2,t}$$

  - It follows from (TA4.18) that p_t is a constant (i.e, depends only on the structural parameters α and σ). This result also holds if $\phi_1 = \phi_2$ but $\Omega_1 \neq \Omega_2$ since in that case $mc_{1,t}^x = \Omega_2/\Omega_1$ (a constant).
- Insight #2: if x_1 and x_2 are be identical from the production side then we can aggregate the model further.

– We know that:

$$\left[\frac{x_{1,t}}{x_{2,t}} =\right] \left(\frac{\alpha}{(1-\alpha)p_t}\right)^{\sigma} = \frac{u_1}{u_2}.$$

so that it follows from (TA4.23) that:

$$u_1 = \frac{1-u_z}{1+\left(\frac{(1-\alpha)p_t}{\alpha}\right)^{\sigma}} = \frac{\alpha^{\sigma}(1-u_z)}{\alpha^{\sigma}+(1-\alpha)^{\sigma}p_t^{\sigma}},$$

$$u_2 = \frac{(1-u_z)\left(\frac{(1-\alpha)p_t}{\alpha}\right)^{\sigma}}{1+\left(\frac{(1-\alpha)p_t}{\alpha}\right)^{\sigma}} = \frac{(1-\alpha)^{\sigma}p_t^{\sigma}(1-u_z)}{\alpha^{\sigma}+(1-\alpha)^{\sigma}p_t^{\sigma}}.$$

– We can thus aggregate total consumption as:

$$\begin{aligned} c_t &\equiv p_t x_{1,t} + x_{2,t} \\ &= p_t u_{1,t}\Omega_1 \kappa_{1,t}^{1-\phi_1} + u_{2,t}\Omega_2\kappa_{2,t}^{1-\phi_2} \\ &= \left[p_t u_{1,t} + u_{2,t}\right]\Omega_x \kappa_{x,t}^{1-\phi} \\ &= \frac{\alpha^{\sigma}p_t + (1-\alpha)^{\sigma}p_t^{\sigma}}{\alpha^{\sigma}+(1-\alpha)^{\sigma}p_t^{\sigma}}(1-u_z)\Omega_x\kappa_{x,t}^{1-\phi} \end{aligned}$$

where $\Omega_x = \Omega_1 = \Omega_2$ and $\kappa_{x,t} = \kappa_{1,t} = \kappa_{2,t}$.

A.1.5.1 Parameterization

- Assert that a steady-state growth equilibrium exists and calibrate it. We calibrate the parameterize the competitive version of the model. We adopt a two-step approach:

  – *Step 1*: parameterize a one-sector version of the model to generate plausible values for κ^*, γ^*, β, Ω, etcetera.

  – *Step 2*: use these plausible value to parameterize the two-sector version of the model .

One-sector model

- Assumptions: $\phi_1 = \phi_2 = \psi = \phi$, $\Omega_1 = \Omega_2 = \Omega_z = \Omega$.

- Dynamic model:

$$\begin{aligned} \frac{K_{t+1}}{\bar{h}_t} &= \frac{1}{1+\beta}\left[\beta w_t\left[1-l_t\right] - \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1+r_{t+1}}\right], \\ \gamma_{t+1} &= \phi_e \frac{l_t^{1-\theta}}{1-\theta}, \\ l_t &\equiv \left[\frac{\lambda\phi_e w_{t+1}}{(1+r_{t+1})w_t}\right]^{1/\theta}, \end{aligned}$$

$$
\begin{aligned}
H_t &= [1+\lambda-l_t]\,\bar{h}_t, \\
w_t &= \phi y_t, \\
r_t &= (1-\phi)\Omega\kappa_t^{-\phi}-\delta, \\
y_t &= \Omega\kappa_t^{1-\phi} \\
\kappa_t &\equiv \frac{K_t}{H_t},
\end{aligned}
$$

where $\gamma_{t+1} \equiv (\bar{h}_{t+1}-\bar{h}_t)/\bar{h}_t$ is the growth rate (of initial human capital). The endogenous variables determined at time t are K_{t+1}, γ_{t+1}, κ_t, H_t, and l_t. The predetermined variables are K_t and $\bar{h}_t$ (so κ_t is a jumping variable).

- Steady-state model:

$$\kappa^* = \frac{w^*}{(1+\beta)(1+\lambda-l^*)}\left[\beta\frac{1-l^*}{1+\gamma^*}-\frac{\lambda}{1+r^*}\right], \tag{A.30}$$

$$\gamma^* = \phi_e\frac{(l^*)^{1-\theta}}{1-\theta}, \tag{A.31}$$

$$l^* \equiv \left[\frac{\lambda\phi_e}{1+r^*}\right]^{1/\theta}, \tag{A.32}$$

$$w^* = \phi y^*, \tag{A.33}$$

$$r^* = (1-\phi)\Omega\,(\kappa^*)^{-\phi}-\delta, \tag{A.34}$$

$$y^* = \Omega\,(\kappa^*)^{1-\phi}, \tag{A.35}$$

where we have used the fact that $(K_{t+1}/\bar{h}_t)^* = ((1+\gamma_{t+1})K_{t+1}/\bar{h}_{t+1})^* = (1+\gamma^*)(1+\lambda-l^*)\kappa^*$ in the first equation.

- We fix the following parameters a priori:
  - Efficiency parameter of human capital: $\phi = 0.75$.
  - Annual physical capital depreciation rate: $\delta_a = 0.06$.
  - Fraction of work time during old-age: $\lambda = 0.5$. In terms of the setting sketched in Figure 1 this means that people retire at age 65.
  - Each adult period is of length $T = 30$ in years.

- We postulate the following targets for the calibration:
  - Annual real interest rate: $r_a = 0.05$.
  - Annual real growth rate: $\gamma_a = 0.025$.
  - The output intensity: $y^*(\equiv Y/H)^* = 1.00$.
  - The time-share of education during youth is $l^* = 0.10$. In terms of Figure 1 this means that people finish college at age 23.

- Each period lasts for 30 years so we find:
  - The interest factor:

$$r^* = (1 + r_a)^T - 1 = 3.3219.$$

  - The growth factor:

$$\gamma^* = (1 + \gamma_a)^T - 1 = 1.0976.$$

  - The depreciation factor:

$$\delta = 1 - (1 - \delta_a)^T = 0.8437.$$

- Since $(r^* + \delta)\kappa^* = (1 - \phi)y^*$ and $y^* = 1$ we find the physical-human capital ratio:

$$\kappa^* = \frac{1 - \phi}{r^* + \delta} = 6.0014 \cdot 10^{-2}.$$

- Since $y^* = \Omega_0(\kappa^*)^{1-\phi} = 1$ we choose Ω_0:

$$\Omega = (\kappa^*)^{\psi - 1} = 2.0204.$$

- The wage rate is:

$$w^* = \phi y^* = 0.75.$$

- From equation (A.32) we find an expression for ϕ_e:

$$\phi_e = \frac{1}{\lambda}(1 + r^*)(l^*)^\theta$$

- Using this expression in combination with equation (A.31) we find the value for θ and ϕ_e:

$$\theta = 1 - \frac{(1 + r^*)l^*}{\lambda(\gamma^*)} = 0.2125, \qquad \phi_e = 5.2998.$$

- Finally we solve equation (A.30) for β:

$$\beta = \frac{\lambda w^*/(1 + r^*) + (1 + \lambda - l^*)\kappa^*}{w^*(1 - l^*)/(1 + \gamma^*) - (1 + \lambda - l^*)\kappa^*} = 0.7182.$$

  This is an feasible value (as $0 < \beta < 1$ is required for discounting).

Two-sector model

- Assumptions: $\phi_1 = \phi_2 = \phi > \psi$, $\Omega_1 = \Omega_2 = \Omega_x \neq \Omega_z$. The investment goods sector is relatively capital-intensive.

- Dynamic model:

$$
\begin{aligned}
(1+\gamma_{t+1})q_t\frac{K_{t+1}}{\bar{h}_{t+1}} &= \frac{1}{1+\beta}\left[\beta w_t\left[1-l_t\right] - \frac{\lambda w_{t+1}}{1+r_{t+1}}\left(1+\gamma_{t+1}\right)\right], \\
\gamma_{t+1} &= \phi_e\frac{l_t^{1-\theta}}{1-\theta}, \\
l_t &\equiv \left[\frac{\lambda\phi_e w_{t+1}}{(1+r_{t+1})w_t}\right]^{1/\theta}, \\
1+r_{t+1} &\equiv \frac{r_{t+1}^k + (1-\delta)q_{t+1}}{q_t}, \\
w_t &= \phi\Omega_x\kappa_{x,t}^{1-\phi}, \\
w_t &= \psi q_t\Omega_z\kappa_{z,t}^{1-\psi}, \\
r_t^k &= (1-\phi)\Omega_x\kappa_{x,t}^{-\phi}, \\
r_t^k &= (1-\psi)q_t\Omega_z\kappa_{z,t}^{-\psi}, \\
\kappa_t &= u_t\kappa_{z,t} + (1-u_t)\kappa_{x,t}, \\
z_t &= \left(\frac{1+\lambda-l_{t+1}}{1+\lambda-l_t}\right)(1+\gamma_{t+1})\kappa_{t+1} - (1-\delta)\kappa_t, \\
\kappa_t &= \frac{1}{1+\lambda-l_t}\frac{K_t}{\bar{h}_t} \\
y_t &= (1-u_t)\Omega_x\kappa_{x,t}^{1-\phi} + q_t u_t\Omega_z\kappa_{z,t}^{1-\psi} \\
z_t &= u_t\Omega_z\kappa_{z,t}^{1-\psi}
\end{aligned}
$$

- Steady-state model:

$$q^*\kappa^* = \frac{w^*}{(1+\beta)(1+\lambda-l^*)}\left[\beta\frac{1-l^*}{1+\gamma^*} - \frac{\lambda}{1+r^*}\right], \tag{A.36}$$

$$\gamma^* = \phi_e\frac{(l^*)^{1-\theta}}{1-\theta}, \tag{A.37}$$

$$l^* \equiv \left[\frac{\lambda\phi_e}{1+r^*}\right]^{1/\theta}, \tag{A.38}$$

$$(r^k)^* = (r^*+\delta)q^*, \tag{A.39}$$

$$w^* = \phi\Omega_x\left(\kappa_x^*\right)^{1-\phi}, \tag{A.40}$$

$$w^* = \psi q^*\Omega_z\left(\kappa_z^*\right)^{1-\psi}, \tag{A.41}$$

$$(r^k)^* = (1-\phi)\Omega_x\left(\kappa_x^*\right)^{-\phi}, \tag{A.42}$$

$$(r^k)^* = (1-\psi)q^*\Omega_z\left(\kappa_z^*\right)^{-\psi}, \tag{A.43}$$

$$\kappa^* = u^*\kappa_z^* + (1-u^*)\kappa_x^*, \tag{A.44}$$

$$z^* = (\gamma^*+\delta)\kappa^*, \tag{A.45}$$

$$y^* = (1-u^*)\Omega_x (\kappa_x^*)^{1-\phi} + q^* u^* \Omega_z (\kappa_z^*)^{1-\psi} \tag{A.46}$$

$$z^* = u^* \Omega_z (\kappa_z^*)^{1-\psi} \tag{A.47}$$

- Targets: key endogenous variables same as in the one-sector model. Hence:

$$y^* = 1, \quad z^* = 0.1165, \quad l^* = 0.1000, \quad \kappa^* = 6.0014 \cdot 10^{-2},$$
$$\gamma^* = 1.0976, \quad w^* = 0.75, \quad r^* = 3.3219, \quad p^* = q^* = 1.$$

- Fixed parameters: β, δ, λ, θ, and ϕ_e.
- We fix $\phi = 0.8$ and choose the following free parameters: ψ, Ω_x, and Ω_z such that all targets are met.
- Steps:
  - By combining (A.40) and (A.42) we find κ_x^*:

$$\kappa_x^* = \frac{1-\phi}{\phi}\left(\frac{w}{r^k}\right)^* = 4.5011 \cdot 10^{-2}.$$

  - From (A.40) and (A.42) we can find the unit-cost function for x. By imposing the target $p^* = 1$ we find:

$$p^* = \left(\frac{w^*}{\phi}\right)^{\phi}\left(\frac{(r^k)^*}{1-\phi}\right)^{1-\phi}\frac{1}{\Omega_x} = 1 \qquad \Leftrightarrow \qquad \Omega_x = 1.7430.$$

  - It follows that total consumption is:

$$x^* = 1 - z^* = (1-u^*)\Omega_x (\kappa_x^*)^{1-\phi} \qquad \Leftrightarrow \qquad 1 - u^* = 0.9424.$$

  - From (A.41) and (A.43) we can find the unit-cost function for z. By imposing the target $q^* = 1$ we can write Ω_z in terms of the unkonw parameter ψ::

$$q^* = \left(\frac{w^*}{\psi}\right)^{\psi}\left(\frac{(r^k)^*}{1-\psi}\right)^{1-\psi}\frac{1}{\Omega_z} = 1 \qquad \Leftrightarrow \qquad \Omega_z = \left(\frac{w^*}{\psi}\right)^{\psi}\left(\frac{(r^k)^*}{1-\psi}\right)^{1-\psi}.$$

  - By combing (A.41) and (A.43) we find κ_z^* in terms of the unkonw parameter ψ:

$$\kappa_z^* = \frac{1-\psi}{\psi}\left(\frac{w}{r^k}\right)^*.$$

  - Investment is:

$$z^* = u^* \left(\frac{w^*}{\psi}\right)^{\psi}\left(\frac{(r^k)^*}{1-\psi}\right)^{1-\psi}\left[\frac{1-\psi}{\psi}\left(\frac{w}{r^k}\right)^*\right]^{1-\psi}$$

$$= \frac{u^* w^*}{\psi} \qquad \Leftrightarrow \qquad \psi = 3.7084 \cdot 10^{-1}.$$

– The value for ψ implies that:

$$\Omega_z = 4.2651.$$

- In summary, the structural parameters for the two-sector model are reported in panel (a) of Table A.1.

A.1.5.2 Visualization: A specific distribution function for η

For the visualizations we use the uniform distribution for η:

- Density and distribution functions:

$$f(\eta) \equiv \frac{1}{\eta_H - \eta_L}, \qquad F(\eta) \equiv \frac{\eta - \eta_L}{\eta_H - \eta_L} \quad (\text{for } \eta_L \leq \eta \leq \eta_H).$$

- Weight:

$$\int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta) = \frac{1-\varepsilon}{(2-\varepsilon)(\eta_H - \eta_L)} \left[\eta_H^{(2-\varepsilon)/(1-\varepsilon)} - \eta_L^{(2-\varepsilon)/(1-\varepsilon)} \right].$$

- Rent-seeking time:

$$e_t(\eta) = \frac{\varepsilon \Pi_{1,t}^m}{W_t \bar{h}_t} \frac{(2-\varepsilon)(\eta_H - \eta_L)}{1-\varepsilon} \frac{\eta^{1/(1-\varepsilon)}}{\eta_H^{(2-\varepsilon)/(1-\varepsilon)} - \eta_L^{(2-\varepsilon)/(1-\varepsilon)}}.$$

- Total rent-seeking effort:

$$E_t = \left[\frac{\varepsilon \Pi_{1,t}^m}{W_t \bar{h}_t} \right]^{\varepsilon} \left[\frac{1-\varepsilon}{(2-\varepsilon)(\eta_H - \eta_L)} \left[\eta_H^{(2-\varepsilon)/(1-\varepsilon)} - \eta_L^{(2-\varepsilon)/(1-\varepsilon)} \right] \right]^{1-\varepsilon}.$$

- Share function:

$$s_t(\eta) = \frac{\eta e_t(\eta)^{\varepsilon}}{E_t} = \frac{(2-\varepsilon)(\eta_H - \eta_L)}{1-\varepsilon} \frac{\eta^{1/(1-\varepsilon)}}{\eta_H^{(2-\varepsilon)/(1-\varepsilon)} - \eta_L^{(2-\varepsilon)/(1-\varepsilon)}}.$$

- Hence:

$$e_t(\eta) = \frac{\varepsilon \Pi_{1,t}^m}{W_t \bar{h}_t} s_t(\eta).$$

- Cumulative share for η in the interval $\eta_0 \leq \eta \leq \eta_1$:

$$S_{\eta_0}^{\eta_1} \equiv \int_{\eta_0}^{\eta_1} s_t(\eta) dF(\eta) = \frac{\eta_1^{(2-\varepsilon)/(1-\varepsilon)} - \eta_0^{(2-\varepsilon)/(1-\varepsilon)}}{\eta_H^{(2-\varepsilon)/(1-\varepsilon)} - \eta_L^{(2-\varepsilon)/(1-\varepsilon)}}.$$

A.1.6 Inequality measures

- The individual lifetime utility function at birth is:

$$\Lambda_t^y(\eta) \equiv \ln c_t^y(\eta) + \beta \ln c_{t+1}^o(\eta)$$

- Aggregate consumption spending by the old (at time t):

$$\frac{P_{V,t} c_t^o}{\bar{h}_t} = \lambda w_t + (1 + r_t) q_t \frac{K_t}{\bar{h}_t}$$

- Aggregate consumption spending by the young (at time t):

$$\frac{P_{V,t} c_t^y}{\bar{h}_t} = \frac{1}{1+\beta} \left[w_t [1 - l_t] + (1 - \varepsilon) \frac{\pi_{1,t}^m}{\bar{h}_t} + \lambda \frac{w_{t+1}(1 + \gamma_{t+1})}{1 + r_{t+1}} \right]$$

- Aggregate saving (equals investment) by the young (at time t):

$$\frac{q_t K_{t+1}}{\bar{h}_t} = \frac{\beta}{1+\beta} \left[w_t [1 - l_t] + (1 - \varepsilon) \frac{\pi_{1,t}^m}{\bar{h}_t} \right] - \frac{\lambda}{1+\beta} \frac{w_{t+1}(1 + \gamma_{t+1})}{1 + r_{t+1}}$$

- Individual consumption spending by the old (at time$t$):

$$\frac{P_{V,t} c_t^o(\eta)}{\bar{h}_t} = \lambda w_t + (1 + r_t) q_t \frac{K_t(\eta)}{\bar{h}_t}$$

- Individual consumption spending by the young (at time t):

$$\frac{P_{V,t} c_t^y(\eta)}{\bar{h}_t} = \frac{1}{1+\beta} \left[w_t [1 - l_t] + s_t(\eta)(1 - \varepsilon) \frac{\pi_{1,t}^m}{\bar{h}_t} + \lambda \frac{w_{t+1}(1 + \gamma_{t+1})}{1 + r_{t+1}} \right],$$

where $s_t(\eta)$ is dependent on the distribution of η:

$$s_t(\eta) \equiv \frac{\eta^{1/(1-\varepsilon)}}{\int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta)}$$

- Aggregate saving (equals investment) by the young (at time t):

$$\frac{q_t K_{t+1}(\eta)}{\bar{h}_t} = \frac{\beta}{1+\beta} \left[w_t [1 - l_t] + s_t(\eta)(1 - \varepsilon) \frac{\pi_{1,t}^m}{\bar{h}_t} \right] - \frac{\lambda}{1+\beta} \frac{w_{t+1}(1 + \gamma_{t+1})}{1 + r_{t+1}},$$

where $K_{t+1}(\eta) = z_t^y(\eta) + k_t^y(\eta)$.

- By combining these results we find for the old:

$$\frac{P_{V,t}[c_t^o(\eta) - c_t^o]}{\bar{h}_t} = (1 + r_t) q_t \frac{K_t(\eta) - K_t}{\bar{h}_t}$$

- For the young we find:

$$\begin{aligned}
\frac{P_{V,t}[c_t^y(\eta) - c_t^y]}{\bar{h}_t} &= \frac{1}{1+\beta}[s_t(\eta) - 1](1-\varepsilon)\frac{\pi_{1,t}^m}{\bar{h}_t} \\
\frac{q_t[K_{t+1}(\eta) - K_{t+1}]}{\bar{h}_t} &= \frac{\beta}{1+\beta}[s_t(\eta) - 1](1-\varepsilon)\frac{\pi_{1,t}^m}{\bar{h}_t}
\end{aligned}$$

- Further notes on welfare computations:
  - Let shock-time be $t = 0$.
  - Following an unanticipated and permanent shock the effect on the shock-time old is computed differently from the way later old-age consumption is treated.
  - For the shock-time old we find that:

    $$\frac{P_{V,t} c_t^o(\eta)}{\bar{h}_t} = \lambda w_t + (1 + r_t) q_t \frac{K_t(\eta)}{\bar{h}_t}$$

    with:

    $$\frac{q_{t-1}[K_t(\eta) - K_t]}{\bar{h}_{t-1}} = \frac{\beta}{1+\beta}[s_{t-1}(\eta) - 1](1-\varepsilon)\frac{\pi_{1,t-1}^m}{\bar{h}_{t-1}}$$

  - So, for example, if the rent-seeking technology is opened up at time t then, obviously we have that $\pi_{1,t-1}^m = 0$ so that we find that $K_t(\eta) = K_t$ (no inequality at all).
  - For the post-shock-time old-age consumption we find that $c_{t+1}(\eta)$ can be linked to $c_t(\eta)$ according to the Euler equation:

    $$\frac{P_{V,t+1}}{1 + r_{t+1}} c_{t+1}^o(\eta) = \beta P_{V,t} c_t^y(\eta),$$

    so that utility at birth for the shock-time young is:

    $$\begin{aligned}
    \Lambda_t^y(\eta) &\equiv \ln c_t^y(\eta) + \beta \ln \left[\frac{\beta(1 + r_{t+1}) P_{V,t} c_t^y(\eta)}{P_{V,t+1}}\right] \\
    &= (1+\beta) \ln c_t^y(\eta) + \beta \ln \left[\frac{\beta(1 + r_{t+1}) P_{V,t}}{P_{V,t+1}}\right]
    \end{aligned}$$

  - Note that the Euler equation in scaled variables is given by:

    $$\frac{c_{t+1}^o(\eta)}{\bar{h}_{t+1}} = \beta \frac{(1 + r_{t+1}) P_{V,t}}{(1 + \gamma_{t+1}) P_{V,t+1}} \frac{c_t^y(\eta)}{\bar{h}_t}.$$

    This expression has been used in the Dynare files.

A.2 Proofs

A.2.1 Useful Result 1

Part (a). For the general model factor demands can be written in real terms (for $i = 1, 2$) as:

$$r_t^k = (1 - \phi_i) MC_i^x(w_t, r_t^k) \Omega_i \kappa_{i,t}^{-\phi_i},$$
$$w_t = \phi_i MC_i^x(w_t, r_t^k) \Omega_i \kappa_{i,t}^{1-\phi_i}.$$

By solving the second equation for $\kappa_{i,t}$ and substituting the resulting expression into the first equation we find:

$$MC_i^x(w_t, r_t^k) = \left(\frac{r_t^k}{1 - \phi_i}\right)^{1-\phi_i} \left(\frac{w_t}{\phi_i}\right)^{\phi_i} \frac{1}{\Omega_i}.$$

In the competitive sector 2 we find $P_{2,t} = MC_2^x(W_t, R_t^k)$ so that:

$$MC_2^x(w_t, r_t^k) = \left(\frac{r_t^k}{1 - \phi_2}\right)^{1-\phi_2} \left(\frac{w_t}{\phi_2}\right)^{\phi_2} \frac{1}{\Omega_2} = 1.$$

For sector 1 we find:

$$MC_1^x(w_t, r_t^k) = \left(\frac{w_t}{\phi_1}\right)^{\phi_1} \left(\frac{r_t^k}{1 - \phi_1}\right)^{1-\phi_1} \frac{1}{\Omega_1}.$$

It follows readily that, for $\phi_1 = \phi_2 = \phi$ and $\Omega_1 = \Omega_2 = \Omega_x$ we obtain:

$$MC_1^x(w_t, r_t^k) = 1, \quad \kappa_{1,t} = \kappa_{2,t}.$$

If $\phi_1 = \phi_2 = \phi$ but $\Omega_1 \neq \Omega_2$ we find that:

$$MC_1^x(w_t, r_t^k) = \frac{\Omega_2}{\Omega_1} \neq 1, \quad \kappa_{1,t} = \kappa_{2,t}.$$

Part (b). For the general model the price is set according to equation (AT2.18). For $\phi_1 = \phi_2 = \phi$ and $\Omega_1 = \Omega_2 = \Omega_x$ we find that $MC_1^x(w_t, r_t^k) = 1$ so that:

$$p_t = \frac{\alpha^\sigma p_t^{1-\sigma} + \sigma(1-\alpha)^\sigma}{(\sigma - 1)(1 - \alpha)^\sigma}.$$

Since α and σ are both time-invariant constants it follows readily that $p_t = p^*$ for all t. To prove that $p > 1$ we note that relative monopoly price satisfies the implicit relationship $\Phi(p^*, \alpha, \sigma) = 0$, with:

$$\Phi(p, \alpha, \sigma) \equiv p + \left(\frac{\alpha}{1 - \alpha}\right)^\sigma \frac{p^{1-\sigma}}{1 - \sigma} - \frac{\sigma}{\sigma - 1}.$$

Table A.1: Structural parameters in the competitive three-sector growth equilibrium

(a) *Coefficients*			
β	time preference parameter	c	0.7182
ρ_a	annual pure rate of time preference (percent)	i	1.1092
λ	proportion of working time in old-age		0.5000
$\phi_1 = \phi_2$	human capital efficiency parameter consumption goods		0.8000
ψ	human capital efficiency parameter investment good	c	0.3708
δ_a	annual capital depreciation rate (percent)		6.0000
δ	capital depreciation factor	i	0.8437
$\Omega_1 = \Omega_2$	scale factor production function consumption goods	c	1.7430
Ω_z	scale factor production function investment good	c	4.2651
θ	curvature parameter of the learning function	c	0.2125
ϕ_e	scale parameter of the learning function	c	5.2998
T	length of each adult period in years		30
(b) *Steady-state equilibrium growth path*			
κ^*	capital intensity:		0.0600
$(K_t/\bar{h}_t)^*$	physical-human capital ratio:		0.0840
l^*	time share of schooling during youth		0.1000
γ^*	growth factor		1.0976
$\gamma_a^* \times 100\%$	annual growth rate (percent)	i	2.5000
r^*	real interest factor		3.3219
$r_a^* \times 100\%$	annual real interest rate (percent)	i	5.0000
w^*	wage rate		0.7500
$(r^k)^*$	rental rate on capital		4.1656
y^*	output intensity		1.0000
$x_1^* = x_2^*$	consumption intensity in each consumption sector		0.4417
z^*	investment intensity		0.1165
q^*	relative price of the investment good		1.0000
$u_1^* = u_2^*$	human capital share in each consumption sector		0.4712
u_z^*	human capital share in the investment sector		0.0576
κ_z^*	capital intensity in the investment sector		0.3055
$\kappa_1^* = \kappa_2^*$	capital intensity in each consumption sector		0.0450

Note The parameters labelled ‘c’ are calibrated as is explained in the text. The ones labelled ‘i’ are implied by other parameters and variables. The remaining parameters are postulated a priori. Note that $\rho_a = \beta^{-1/T} - 1$, $r_a^* = (1+r^*)^{1/T} - 1$, $\gamma_a^* = (1+\gamma^*)^{1/T} - 1$, and $\delta = 1 - (1-\delta_a)^T$.

Table A.2: Rent-seeking and growth in the three-sector model (TY scenario)

$$(1+\gamma_{t+1})q_t\frac{K_{t+1}}{\bar{h}_{t+1}} = \frac{1}{1+\beta}\left[\beta(1-\varepsilon)\frac{\pi^m_{1,t}}{\bar{h}_t} + \beta w_t\left(1-l_t\right) - \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1+r_{t+1}}\right] \quad \text{(AT2.1)}$$

$$\frac{\pi^m_{1,t}}{\bar{h}_t} = \frac{\Xi_t}{1+\beta-(1-\varepsilon)\Xi_t}\left[w_t\left(1-l_t\right) + \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1+r_{t+1}}\right] + \frac{(1+\beta)\Xi_t}{1+\beta-(1-\varepsilon)\Xi_t}\left[\lambda w_t + \left(\left(1-\delta\right)q_t + r^k_t\right)\frac{K_t}{\bar{h}_t}\right] \quad \text{(AT2.2)}$$

$$w_t\bar{e}_t = \varepsilon\frac{\pi^m_{1,t}}{\bar{h}_t} \quad \text{(AT2.3)}$$

$$\gamma_{t+1} = \phi_e\frac{l_t^{1-\theta}}{1-\theta} \quad \text{(AT2.4)}$$

$$l_t^\theta \equiv \frac{\lambda\phi_e w_{t+1}}{(1+r_{t+1})w_t} \quad \text{(AT2.5)}$$

$$1+r_{t+1} \equiv \frac{r^k_{t+1}+(1-\delta)q_{t+1}}{q_t} \quad \text{(AT2.6)}$$

$$w_t = \phi_1 mc^x_{1,t}\Omega_1\kappa_{1,t}^{1-\phi_1} = \phi_2\Omega_2\kappa_{2,t}^{1-\phi_2} = \psi q_t\Omega_z\kappa_{z,t}^{1-\psi} \quad \text{(AT2.7)-(AT2.9)}$$

$$r^k_t = (1-\phi_1)mc^x_{1,t}\Omega_1\kappa_{1,t}^{-\phi_1} = (1-\phi_2)\Omega_2\kappa_{2,t}^{-\phi_2} = (1-\psi)q_t\Omega_z\kappa_{z,t}^{-\psi} \quad \text{(AT2.10)-(AT2.12)}$$

$$\kappa_t = u_{1,t}\kappa_{1,t} + u_{2,t}\kappa_{2,t} + u_{z,t}\kappa_{z,t} \quad \text{(AT2.13)}$$

$$z_t = \left(\frac{1+\lambda-\bar{e}_{t+1}-l_{t+1}}{1+\lambda-\bar{e}_t-l_t}\right)(1+\gamma_{t+1})\kappa_{t+1} - (1-\delta)\kappa_t \quad \text{(AT2.14)}$$

$$\kappa_t = \frac{1}{1+\lambda-\bar{e}_t-l_t}\frac{K_t}{\bar{h}_t} \quad \text{(AT2.15)}$$

$$y_t = p_t x_{1,t} + x_{2,t} + q_t z_t \quad \text{(AT2.16)}$$

$$\Xi_t \equiv \frac{\alpha^\sigma p_t^{1-\sigma}}{\alpha^\sigma p_t^{1-\sigma}+\sigma(1-\alpha)^\sigma} \quad \text{(AT2.17)}$$

$$p_t = \frac{\alpha^\sigma p_t^{1-\sigma}+\sigma(1-\alpha)^\sigma}{(\sigma-1)\left(1-\alpha\right)^\sigma}mc^x_{1,t} \quad \text{(AT2.18)}$$

$$p_{1,t}x_{1,t} = \frac{\alpha^\sigma p_t^{1-\sigma}}{\alpha^\sigma p_t^{1-\sigma}+(1-\alpha)^\sigma}\frac{1}{1+\lambda-\bar{e}_t-l_t}\left[\frac{1}{1+\beta}\left((1-\varepsilon)\frac{\pi^m_{1,t}}{\bar{h}_t} + w_t\left(1-l_t\right) + \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1+r_{t+1}}\right) + \lambda w_t + \left(\left(1-\delta\right)q_t + r^k_t\right)\frac{K_t}{\bar{h}_t}\right] \quad \text{(AT2.19)}$$

$$x_{1,t} = u_{1,t}\Omega_1\kappa_{1,t}^{1-\phi_1} \quad \text{(AT2.20)}$$

$$x_{2,t} = u_{2,t}\Omega_2\kappa_{2,t}^{1-\phi_2} \quad \text{(AT2.21)}$$

$$z_t = u_{z,t}\Omega_z\kappa_{z,t}^{1-\psi} \quad \text{(AT2.22)}$$

$$1 = u_{1,t} + u_{2,t} + u_{z,t} \quad \text{(AT2.23)}$$

Notes The endogenous variables are $K_{t+1}/\bar{h}_{t+1}$, γ_{t+1}, $\bar{e}_t$, $\pi^m_{1,t}/\bar{h}_t$, l_t, r_t, q_t, r^k_t, w_t, $x_{1,t} \equiv X_{1,t}/H_t$, $x_{2,t} \equiv X_{2,t}/H_t$, $z_t \equiv Z_t/H_t$, $u_{1,t} \equiv H_{1,t}/H_t$, $u_{2,t} \equiv H_{2,t}/H_t$, $u_{z,t} \equiv H_{z,t}/H_t$, $\kappa_t \equiv K_t/H_t$, $\kappa_{1,t} \equiv K_{1,t}/H_{1,t}$, $\kappa_{2,t} \equiv K_{2,t}/H_{2,t}$, $\kappa_{z,t} \equiv K_{z,t}/H_{z,t}$, $mc^x_{1,t}$, Ξ_t, p_t, and $y_t \equiv Y_t/H_t$. Of these, only $K_t/\bar{h}_t$ is predetermined at time t.

By differentiation we find that:

$$\Phi_p \equiv \frac{\partial \Phi(p,\alpha,\sigma)}{\partial p} = \left[1 + \left(\frac{\alpha}{(1-\alpha)p}\right)^{\sigma}\right] > 0,$$

so $\Phi(p,\alpha,\sigma)$ is increasing. We also find that:

$$\Phi(1,\alpha,\sigma) = -\frac{1}{\sigma-1}\left[1 + \left(\frac{\alpha}{1-\alpha}\right)^{\sigma}\right] < 0,$$

from which it follows that $p > 1$.

Part (c). By differentiating $\Phi(p,\alpha,\sigma)$ with respect to α we find:

$$\Phi_\alpha \equiv \frac{\partial \Phi(p,\alpha,\sigma)}{\partial \alpha} = -\frac{\sigma}{\sigma-1}\left(\frac{\alpha}{1-\alpha}\right)^{\sigma-1}\frac{p^{1-\sigma}}{(1-\alpha)^2} < 0.$$

It follows readily that $\partial p^*/\partial\alpha = -\Phi_\alpha/\Phi_p > 0$.

Part (d). By differentiating $\Phi(p,\alpha,\sigma)$ with respect σ we find:

$$\Phi_\sigma \equiv \frac{\partial \Phi(p,\alpha,\sigma)}{\partial \sigma} = \frac{1}{(\sigma-1)^2}\left[1 + \left(\frac{\alpha}{1-\alpha}\right)^{\sigma} p^{1-\sigma}\left[1+(\sigma-1)\ln p\right]\right] - \frac{p^{1-\sigma}}{\sigma-1}\left(\frac{\alpha}{1-\alpha}\right)^{\sigma}\ln\left(\frac{\alpha}{1-\alpha}\right).$$

The first term on the right-hand side is positive but the second term is negative for $\alpha > \frac{1}{2}$. Evaluated at $\alpha = \frac{1}{2}$ this term vanishes and $\Phi_\sigma > 0$. It then follows readily that $\partial p/\partial\sigma = -\Phi_\sigma/\Phi_p < 0$.

Part (e). The proportionality factor Ξ_t is defined in equation (AT2.17). Its time-invariance, $\Xi_t = \Xi^*$, follows immediately from the fact that $p_t = p^*$ for all t.

A.2.2 Useful Result ??

???? To be added

A.2.3 Analytical results

- The competitive steady-state growth model is listed in Table 2 in the paper. Equations (T2.1)–(T2.10) and (T2.12) can be linearized around the initial steady state to obtain:

$$\tilde{\gamma}^* = (1-\theta)\tilde{l}^* \tag{TAx.1}$$

$$\tilde{l}^* = -\frac{r^*}{\theta(1+r^*)}\,\tilde{r}^* \tag{TAx.2}$$

$$(\tilde{r}^k)^* = \frac{r^*}{r^*+\delta}\,\tilde{r}^* + \tilde{q}^* \tag{TAx.3}$$

$$\tilde{w}^* = (1-\phi)\tilde{\kappa}_x^* \tag{TAx.4}$$

$$\tilde{w}^* = \tilde{q}^* + (1-\psi)\tilde{\kappa}_z^* \tag{TAx.5}$$

$$(\tilde{r}^k)^* = -\phi\tilde{\kappa}_x^* \tag{TAx.6}$$

$$(\tilde{r}^k)^* = \tilde{q}^* - \psi\tilde{\kappa}_z^* \tag{TAx.7}$$

$$\tilde{\kappa}^* = \frac{u_z^*\kappa_z^*}{\kappa^*}\,\tilde{\kappa}_z + \frac{(1-u_z^*)\kappa_x^*}{\kappa^*}\,\tilde{\kappa}_x + \frac{u_z^*(\kappa_z^*-\kappa_x^*)}{\kappa^*}\,\tilde{u}_z \tag{TAx.8}$$

$$\tilde{z}^* = \frac{\gamma^*}{\gamma^*+\delta}\,\tilde{\gamma}^* + \tilde{\kappa}^* \tag{TAx.9}$$

$$\tilde{z}^* = \tilde{u}_z^* + (1-\psi)\tilde{\kappa}_z^* \tag{TAx.12}$$

- The dimensionality of the model can be reduced quite substantially.
  - Equations (TAx.4) and (TAx.6) imply:

$$\tilde{w}^* - (\tilde{r}^k)^* = \tilde{\kappa}_x^*$$

  - Equations (TAx.5) and (TAx.7) imply:

$$\tilde{w}^* - (\tilde{r}^k)^* = \tilde{\kappa}_z^*$$

  - Hence:

$$\tilde{\kappa}_x^* = \tilde{\kappa}_z^*$$

  - Equations (TAx.4) and (TAx.5) then imply:

$$\tilde{q}^* = (\psi-\phi)\tilde{\kappa}_x^*$$

  - Equations (TAx.3) and (TAx.6) then imply:

$$\tilde{r}^* = -\frac{\psi(r^*+\delta)}{r^*}\tilde{\kappa}_x^*$$

- Equations (TAx.1) and (TAx.2) then imply:

$$\tilde{\gamma}^* = (1-\theta)\tilde{l}^* = \frac{1-\theta}{\theta}\frac{\psi(r^*+\delta)}{1+r^*}\tilde{\kappa}_x^*$$

- We are left with a system, determining $(\tilde{\kappa}_x, \tilde{u}_z, \tilde{z}^*)$ in terms of $\tilde{\kappa}^*$:

$$\begin{aligned}
\tilde{\kappa}^* &= \tilde{\kappa}_x^* + \frac{u_z^*(\kappa_z^* - \kappa_x^*)}{\kappa^*}\tilde{u}_z \\
\tilde{z}^* &= \frac{\gamma^*}{\gamma^*+\delta}\frac{1-\theta}{\theta}\frac{\psi(r^*+\delta)}{1+r^*}\tilde{\kappa}_x^* + \tilde{\kappa}^* \\
\tilde{z}^* &= \tilde{u}_z^* + (1-\psi)\tilde{\kappa}_x^*
\end{aligned}$$

- In matrix notation:

$$\Delta \begin{bmatrix} \tilde{\kappa}_x^* \\ \tilde{u}_z \\ \tilde{z}^* \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tilde{\kappa}^*$$

where Δ is given by:

$$\Delta \equiv \begin{bmatrix} 1 & \frac{u_z^*(\kappa_z^*-\kappa_x^*)}{\kappa^*} & 0 \\ -\frac{\gamma^*}{\gamma^*+\delta}\frac{1-\theta}{\theta}\frac{\psi(r^*+\delta)}{1+r^*} & 0 & 1 \\ 1-\psi & 1 & -1 \end{bmatrix} \equiv \begin{bmatrix} 1 & \delta_{12} & 0 \\ -\delta_{21} & 0 & 1 \\ 1-\psi & 1 & -1 \end{bmatrix}.$$

- We easily find:

  - The determinant:

$$\begin{aligned}
|\Delta| &= \delta_{12}\,(1-\psi-\delta_{21}) - 1 \\
&= \frac{u_z^*(\kappa_z^*-\kappa_x^*)}{\kappa^*}\left(1-\psi-\frac{\gamma^*}{\gamma^*+\delta}\frac{1-\theta}{\theta}\frac{\psi(r^*+\delta)}{1+r^*}\right) - 1.
\end{aligned}$$

  - The inverse:

$$\begin{aligned}
\Delta^{-1} &= -\frac{1}{|\Delta|}\begin{bmatrix} 1 & -\delta_{12} & -\delta_{12} \\ \psi+\delta_{21}-1 & 1 & 1 \\ \delta_{21} & (\psi-1)\,\delta_{12}+1 & -\delta_{12}\delta_{21} \end{bmatrix} \\
&= -\frac{1}{|\Delta|}\begin{bmatrix} 1 & -\frac{u_z^*(\kappa_z^*-\kappa_x^*)}{\kappa^*} & -\frac{u_z^*(\kappa_z^*-\kappa_x^*)}{\kappa^*} \\ \psi+\frac{\gamma^*}{\gamma^*+\delta}\frac{1-\theta}{\theta}\frac{\psi(r^*+\delta)}{1+r^*}-1 & 1 & 1 \\ \frac{\gamma^*}{\gamma^*+\delta}\frac{1-\theta}{\theta}\frac{\psi(r^*+\delta)}{1+r^*} & (\psi-1)\frac{u_z^*(\kappa_z^*-\kappa_x^*)}{\kappa^*}+1 & -\frac{u_z^*(\kappa_z^*-\kappa_x^*)}{\kappa^*}\frac{\gamma^*}{\gamma^*+\delta}\frac{1-\theta}{\theta}\frac{\psi(r^*+\delta)}{1+r^*} \end{bmatrix}
\end{aligned}$$

– The solution:

$$\Delta \left[\begin{array}{c} \tilde{\kappa}_x^* \\ \tilde{u}_z \\ \tilde{z}^* \end{array} \right] = -\frac{1}{|\Delta|} \left[\begin{array}{c} 1-\delta_{12} \\ \psi+\delta_{21} \\ \delta_{21}+(\psi-1)\,\delta_{12}+1 \end{array} \right] \tilde{\kappa}^*$$

$$= -\frac{1}{|\Delta|} \left[\begin{array}{c} 1-\frac{u_z^*(\kappa_z^*-\kappa_x^*)}{\kappa^*} \\ \psi+\frac{\gamma^*}{\gamma^*+\delta}\frac{1-\theta}{\theta}\frac{\psi(r^*+\delta)}{1+r^*} \\ \frac{\gamma^*}{\gamma^*+\delta}\frac{1-\theta}{\theta}\frac{\psi(r^*+\delta)}{1+r^*}+(\psi-1)\,\frac{u_z^*(\kappa_z^*-\kappa_x^*)}{\kappa^*}+1 \end{array} \right] \tilde{\kappa}^*$$

- For future use we note that:

$$\kappa_z^* - \kappa_x^* = \frac{\phi-\psi}{\phi\psi}\left(\frac{w}{r^k}\right)^*$$

A.3 Alternative timing: proceeds to the old

- Rent-seeking activities during youth give a payoff during old-age.
- Budget constraint during youth:

$$P_{V,t}c_t^y(\eta) + Q_t\left[z_t^y(\eta)+k_t^y(\eta)\right] = W_t h_t^y(\eta)\left[1-e_t(\eta)-l_t(\eta)\right].$$

- Budget constraint during old-age:

$$P_{V,t+1}c_{t+1}^o(\eta) = \lambda W_{t+1}h_{t+1}^o(\eta) + s_t(\eta)\Pi_{1,t+1}^m + \left[(1-\delta)Q_{t+1}+R_{t+1}^k\right]\left[z_t^y(\eta)+k_t^y(\eta)\right].$$

- Consolidated budget constraint in nominal terms:

$$P_{V,t}c_t^y(\eta) + \frac{P_{V,t+1}c_{t+1}^o(\eta)}{1+R_{t+1}^n} = HW_t^y, \tag{A.48}$$

where human wealth during youth is:

$$HW_t^y \equiv W_t h_t^y(\eta)\left[1-e_t(\eta)-l_t(\eta)\right] + \frac{\lambda W_{t+1}h_{t+1}^o(\eta)+s_t(\eta)\Pi_{1,t+1}^m}{1+R_{t+1}^n}.$$

- Lagrangian:

$$\mathcal{L}_t^y \equiv \ln c_t^y(\eta) + \beta\ln c_{t+1}^o(\eta) + \mu_t\left[\frac{s_t(\eta)\Pi_{1,t+1}^m}{1+R_{t+1}^n} + W_t\bar{h}_t\left[1-e_t(\eta)-l_t(\eta)\right]\right.$$
$$\left. + \frac{\lambda W_{t+1}}{1+R_{t+1}^n}\bar{h}_t\left[1+\phi_e\frac{l_t(\eta)^{1-\theta}}{1-\theta}\right] - P_{V,t}c_t^y(\eta) - \frac{P_{V,t+1}c_{t+1}^o(\eta)}{1+R_{t+1}^n}\right].$$

- First-order conditions:

$$
\begin{aligned}
\frac{\partial \mathcal{L}_t^y}{\partial c_t^y(\eta)} &= \frac{1}{c_t^y(\eta)} - \mu_t P_{V,t} = 0, \\
\frac{\partial \mathcal{L}_t^y}{\partial c_{t+1}^o(\eta)} &= \frac{\beta}{c_{t+1}^o(\eta)} - \frac{\mu_t P_{V,t+1}}{1+R_{t+1}^n} = 0, \\
\frac{\partial \mathcal{L}_t^y}{\partial l_t(\eta)} &= \mu_t \left[-W_t + \frac{W_{t+1}}{1+R_{t+1}^n} \lambda \phi_e l_t(\eta)^{-\theta} \right] \bar{h}_t = 0, \\
\frac{\partial \mathcal{L}_t^y}{\partial e_t(\eta)} &= \mu_t \left[\frac{\Pi_{1,t+1}^m}{1+R_{t+1}^n} \frac{\partial s_t(\eta)}{\partial e_t(\eta)} - W_t \bar{h}_t \right] = 0.
\end{aligned}
$$

- Substituting the first two expressions into the life-time budget constraint (A.48) we find:

$$
\begin{aligned}
P_{V,t} c_t^y(\eta) &= \frac{1}{1+\beta} HW_t^y(\eta), \\
\frac{P_{V,t+1} c_{t+1}^o(\eta)}{1+R_{t+1}^n} &= \frac{\beta}{1+\beta} HW_t^y(\eta).
\end{aligned}
$$

- Saving during youth:

$$
Q_t \left[z_t^y(\eta) + k_t^y(\eta) \right] = W_t h_t^y(\eta) \left[1 - e_t(\eta) - l_t(\eta) \right] - P_{V,t} c_t^y(\eta).
$$

- Simplify:

$$
Q_t \left[z_t^y(\eta) + k_t^y(\eta) \right] = \frac{\beta}{1+\beta} W_t h_t^y(\eta) \left[1 - e_t(\eta) - l_t(\eta) \right] - \frac{1}{1+\beta} \frac{\lambda W_{t+1} h_{t+1}^o(\eta) + s_t(\eta) \Pi_{1,t+1}^m}{1+R_{t+1}^n}.
$$

- For the success function $s_t(\eta) = \eta e_t(\eta)^{\varepsilon}/R_t$ we find:

$$
\varepsilon \frac{\Pi_{1,t+1}^m}{1+R_{t+1}^n} \frac{\eta e_t(\eta)^{\varepsilon-1}}{R_t} = W_t \bar{h}_t \qquad \Leftrightarrow \qquad e_t(\eta) = \left[\frac{\varepsilon}{W_t \bar{h}_t} \frac{\Pi_{1,t+1}^m}{1+R_{t+1}^n} \frac{\eta}{R_t} \right]^{1/(1-\varepsilon)}.
$$

- It follows that total rent-seeking effort E_t and wasted labour $\bar{e}_t$ amount to:

$$
\begin{aligned}
E_t &\equiv \int_{\eta_L}^{\eta_H} \eta e_t(\eta)^{\varepsilon} dF(\eta) = \left[\frac{\varepsilon \Pi_{1,t+1}^m}{W_t \bar{h}_t E_t (1+R_{t+1}^n)} \right]^{\varepsilon/(1-\varepsilon)} \int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta), \\
\bar{e}_t &\equiv \int_{\eta_L}^{\eta_H} e_t(\eta) dF(\eta) = \left[\frac{\varepsilon \Pi_{1,t+1}^m}{W_t \bar{h}_t E_t (1+R_{t+1}^n)} \right]^{1/(1-\varepsilon)} \int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta).
\end{aligned}
$$

- Solving for R_t gives:

$$
E_t = \left[\frac{\varepsilon \Pi_{1,t+1}^m}{W_t \bar{h}_t (1+R_{t+1}^n)} \right]^{\varepsilon} \left[\int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta) \right]^{1-\varepsilon}.
$$

- Solving for $e_t(\eta)$ gives:

$$e_t(\eta) = \frac{\varepsilon \Pi^m_{1,t+1}}{W_t \bar{h}_t (1 + R^n_{t+1})} \frac{\eta^{1/(1-\varepsilon)}}{\int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta)}$$

- Solving for $\bar{e}_t$ gives:

$$\bar{e}_t = \frac{\varepsilon \Pi^m_{1,t+1}}{W_t \bar{h}_t (1 + R^n_{t+1})} \tag{A.49}$$

- We need to find an expression for $\Pi^m_{1,t+1}$.
- Demand in sector 1 originates from the young and the old.
  - Young demand for good 1:

$$X^y_{1,t} = \frac{\alpha^\sigma P^{-\sigma}_{1,t}}{\alpha^\sigma P^{1-\sigma}_{1,t} + (1-\alpha)^\sigma P^{1-\sigma}_{2,t}} \frac{HW^y_t}{1+\beta}.$$

  - Old demand for good 1:

$$X^o_{1,t} = \frac{\alpha^\sigma P^{-\sigma}_{1,t}}{\alpha^\sigma P^{1-\sigma}_{1,t} + (1-\alpha)^\sigma P^{1-\sigma}_{2,t}} I^o_t,$$
$$I^o_t = \lambda W_t h^o_t + \Pi^m_{1,t} + \left[(1-\delta) Q_t + R^k_t\right] K_t.$$

  - Total demand is thus:

$$X_{1,t} = \frac{\alpha^\sigma P^{-\sigma}_{1,t}}{\alpha^\sigma P^{1-\sigma}_{1,t} + (1-\alpha)^\sigma P^{1-\sigma}_{2,t}} \left[\frac{HW^y_t}{1+\beta} + I^o_t\right].$$

- Aggregate profit equals:

$$\Pi^m_{1,t} = \Xi_t \left[\frac{HW^y_t}{1+\beta} + I^o_t\right]. \tag{A.50}$$

  where Ξ_t is defined above (see (A.26)).

- We find (after using (A.49)) that:

$$\frac{HW^y_t}{1+\beta} + I^o_t = \frac{1}{1+\beta} \left(W_t \bar{h}_t \left[1 - l_t\right] + \frac{\lambda W_{t+1} \bar{h}_t (1 + \gamma_{t+1}) + (1-\varepsilon) \Pi^m_{1,t+1}}{1 + R^n_{t+1}} \right)$$
$$+ \lambda W_t \bar{h}_t + \Pi^m_{1,t} + \left[(1-\delta) Q_t + R^k_t\right] K_t. \tag{A.51}$$

  - Current profit depends in part on itself because **old** agents consume it
  - Current profit depends in part on expected future profit because these form part of human wealth of **young** agents

- By solving (A.50) and (A.51) for $\Pi^m_{1,t}$ and $\frac{HW^y_t}{1+\beta} + I^o_t$ we find:

$$\Pi^m_{1,t} = \frac{\Xi_t}{(1-\Xi_t)(1+\beta)}\Bigg[W_t\bar{h}_t\left[1-l_t\right] + \frac{\lambda W_{t+1}\bar{h}_t(1+\gamma_{t+1}) + (1-\varepsilon)\Pi^m_{1,t+1}}{1+R^n_{t+1}}$$
$$+ (1+\beta)\left[\lambda W_t\bar{h}_t + \left[(1-\delta)\,Q_t + R^k_t\right]K_t\right]\Bigg].$$

and:

$$\frac{HW^y_t}{1+\beta} + I^o_t = \frac{1}{(1-\Xi_t)(1+\beta)}\Bigg[W_t\bar{h}_t\left[1-l_t\right] + \frac{\lambda W_{t+1}\bar{h}_t(1+\gamma_{t+1}) + (1-\varepsilon)\Pi^m_{1,t+1}}{1+R^n_{t+1}}$$
$$+ (1+\beta)\left[\lambda W_t\bar{h}_t + \left[(1-\delta)\,Q_t + R^k_t\right]K_t\right]\Bigg].$$

- The key equations of the model with alternative timing are gathered in Table A.3.
- The various scenarios are reported in Table A.4. Rent seeking destroys economic growth.

A.3.1 Verify Walras Law

- Spending at time t:

$$P_{V,t}\left[c^y_t + c^o_t\right] + Q_t\left[Z_t + (1-\delta)K_t\right] = W_t\bar{h}_t\left[1-\bar{e}_t-l_t\right] + \Pi^m_{1,t} + \lambda W_t h^o_t + \left[(1-\delta)Q_t + R^k_t\right]K_t.$$

- The old sell the remaining capital to the young so:

$$P_{V,t}\left[c^y_t + c^o_t\right] + Q_t Z_t = W_t\bar{h}_t\left[1-\bar{e}_t-l_t\right] + \Pi^m_{1,t} + \lambda W_t h^o_t + R^k_t K_t.$$

- But $P_{V,t}\left[c^y_t + c^o_t\right] = P_{1,t}X_{1,t} + P_{2,t}X_{2,t}$ and $H_t = \bar{h}_t\left[1-\bar{e}_t-l_t\right] + \lambda h^o_t$ so we get:

$$P_{1,t}X_{1,t} + P_{2,t}X_{2,t} + Q_t Z_t = W_t H_t + \Pi^m_{1,t} + R^k_t K_t.$$

- But $\Pi^m_{1,t} = (P_{1,t} - MC^x_1(W_t,R^k_t))X_{1,t}$, $P_{2,t} = MC^x_2(W_t,R^k_t)$, and $Q_t = MC^z(W_t,R^k_t)$ so we get:

$$MC^x_1(W_t,R^k_t)X_{1,t} + MC^x_2(W_t,R^k_t)X_{2,t} + MC^z(W_t,R^k_t)Z_t = W_t H_t + R^k_t K_t.$$

Right-hand side: total factor income. Left-hand side: total spending on consumption and investment goods evaluated at the true marginal cost of producing these goods.

A.3.2 Checking market equilibrium conditions

- Market for good 1 (demand and supply):

$$
\begin{aligned}
p_{1,t}\frac{X_{1,t}}{\bar{h}_t} &= \frac{\alpha^{\sigma}p_{1,t}^{1-\sigma}}{\alpha^{\sigma}p_{1,t}^{1-\sigma}+(1-\alpha)^{\sigma}}\left[\frac{1}{1+\beta}\left(w_t\left[1-l_t\right]+\frac{1+\gamma_{t+1}}{1+r_{t+1}}\left[\lambda w_{t+1}+(1-\varepsilon)\frac{\pi^{m}_{1,t+1}}{\bar{h}_{t+1}}\right]\right)\right.\\
&\quad\left.+\lambda w_t+\frac{\pi^{m}_{1,t}}{\bar{h}_t}+\left[(1-\delta)\,q_t+r^{k}_t\right]\frac{K_t}{\bar{h}_t}\right].\\
\frac{X_{1,t}}{H_t} &= \Omega_1 u_{1,t}\kappa_{1,t}^{1-\phi_1}.
\end{aligned}
$$

- Market for good 2 (demand and supply):

$$
\begin{aligned}
\frac{X_{2,t}}{\bar{h}_t} &= \frac{(1-\alpha)^{\sigma}}{\alpha^{\sigma}p_{1,t}^{1-\sigma}+(1-\alpha)^{\sigma}}\left[\frac{1}{1+\beta}\left(w_t\left[1-l_t\right]+\frac{1+\gamma_{t+1}}{1+r_{t+1}}\left[\lambda w_{t+1}+(1-\varepsilon)\frac{\pi^{m}_{1,t+1}}{\bar{h}_{t+1}}\right]\right)\right.\\
&\quad\left.+\lambda w_t+\frac{\pi^{m}_{1,t}}{\bar{h}_t}+\left[(1-\delta)\,q_t+r^{k}_t\right]\frac{K_t}{\bar{h}_t}\right],\\
\frac{X_{2,t}}{H_t} &= \Omega_2 u_{2,t}\kappa_{2,t}^{1-\phi_2}.
\end{aligned}
$$

- Market for investment goods (demand and supply):

$$
\begin{aligned}
q_t\frac{Z_t}{\bar{h}_t} &= \frac{\beta}{1+\beta}w_t\left[1-l_t\right]\Big] - \frac{1}{1+\beta}\frac{1+\gamma_{t+1}}{1+r_{t+1}}\left[\lambda w_{t+1}+(1+\beta\varepsilon)\frac{\pi^{m}_{1,t+1}}{\bar{h}_{t+1}}\right]-q_t(1-\delta)\frac{K_t}{\bar{h}_t},\\
\frac{Z_t}{H_t} &= u_{z,t}\Omega_z\kappa_{z,t}^{1-\psi}.
\end{aligned}
$$

- Aggregate output:

$$
\frac{Y_t}{H_t} = p_t\frac{X_{1,t}}{H_t}+\frac{X_{2,t}}{H_t}+q_t\frac{Z_t}{H_t}.
$$

Table A.3: Features of the steady-state growth path (TO scenario)

$$(1+\gamma_{t+1})q_t\frac{K_{t+1}}{\bar{h}_{t+1}} = \frac{1}{1+\beta}\left[\beta w_t\left(1-l_t\right) - \frac{1+\gamma_{t+1}}{1+r_{t+1}}\left(\lambda w_{t+1} + (1+\beta\varepsilon)\frac{\pi^m_{1,t+1}}{\bar{h}_{t+1}}\right)\right] \quad \text{(AT3.1)}$$

$$\frac{\pi^m_{1,t}}{\bar{h}_t} = \frac{\Xi_t}{(1-\Xi_t)(1+\beta)}\left[w_t\left(1-l_t\right) + \frac{1+\gamma_{t+1}}{1+r_{t+1}}\left(\lambda w_{t+1} + (1-\varepsilon)\frac{\pi^m_{1,t+1}}{\bar{h}_{t+1}}\right)\right]$$
$$+ \frac{\Xi_t}{1-\Xi_t}\left[\lambda w_t + \left(\left(1-\delta\right)q_t + r^k_t\right)\frac{K_t}{\bar{h}_t}\right] \quad \text{(AT3.2)}$$

$$w_t\bar{e}_t = \varepsilon\frac{1+\gamma_{t+1}}{1+r_{t+1}}\frac{\pi^m_{1,t+1}}{\bar{h}_{t+1}} \quad \text{(AT3.3)}$$

$$\gamma_{t+1} = \phi_e\frac{l_t^{1-\theta}}{1-\theta} \quad \text{(AT3.4)}$$

$$l_t^\theta \equiv \frac{\lambda\phi_e w_{t+1}}{(1+r_{t+1})w_t} \quad \text{(AT3.5)}$$

$$1+r_{t+1} \equiv \frac{r^k_{t+1} + (1-\delta)q_{t+1}}{q_t} \quad \text{(AT3.6)}$$

$$w_t = \phi_1 mc^x_{1,t}\Omega_1\kappa_{1,t}^{1-\phi_1} = \phi_2\Omega_2\kappa_{2,t}^{1-\phi_2} = \psi q_t\Omega_z\kappa_{z,t}^{1-\psi} \quad \text{(AT3.7)-(AT3.9)}$$

$$r^k_t = (1-\phi_1)mc^x_{1,t}\Omega_1\kappa_{1,t}^{-\phi_1} = (1-\phi_2)\Omega_2\kappa_{2,t}^{-\phi_2} = (1-\psi)q_t\Omega_z\kappa_{z,t}^{-\psi} \quad \text{(AT3.10)-(AT3.12)}$$

$$\kappa_t = u_{1,t}\kappa_{1,t} + u_{2,t}\kappa_{2,t} + u_{z,t}\kappa_{z,t} \quad \text{(AT3.13)}$$

$$z_t = \left(\frac{1+\lambda-\bar{e}_{t+1}-l_{t+1}}{1+\lambda-\bar{e}_t-l_t}\right)(1+\gamma_{t+1})\kappa_{t+1} - (1-\delta)\kappa_t \quad \text{(AT3.14)}$$

$$\kappa_t = \frac{1}{1+\lambda-\bar{e}_t-l_t}\frac{K_t}{\bar{h}_t} \quad \text{(AT3.15)}$$

$$y_t = p_t x_{1,t} + x_{2,t} + q_t z_t \quad \text{(AT3.16)}$$

$$\Xi_t \equiv \frac{\alpha^\sigma p_t^{1-\sigma}}{\alpha^\sigma p_t^{1-\sigma} + \sigma(1-\alpha)^\sigma} \quad \text{(AT3.17)}$$

$$p_t = \frac{\alpha^\sigma p_t^{1-\sigma} + \sigma(1-\alpha)^\sigma}{(\sigma-1)\left(1-\alpha\right)^\sigma}mc^x_{1,t} \quad \text{(AT3.18)}$$

$$p_t x_{1,t} = \frac{\alpha^\sigma p_t^{1-\sigma}}{\alpha^\sigma p_t^{1-\sigma} + (1-\alpha)^\sigma}\frac{1}{1+\lambda-\bar{e}_t-l_t}\left[\frac{1}{1+\beta}\left(w_t\left(1-l_t\right) + \frac{1+\gamma_{t+1}}{1+r_{t+1}}\left(\lambda w_{t+1} + (1-\varepsilon)\frac{\pi^m_{1,t+1}}{\bar{h}_{t+1}}\right)\right)\right.$$
$$\left. + \lambda w_t + \frac{\pi^m_{1,t}}{\bar{h}_t} + \left(\left(1-\delta\right)q_t + r^k_t\right)\frac{K_t}{\bar{h}_t}\right] \quad \text{(AT3.19)}$$

$$x_{1,t} = u_{1,t}\Omega_1\kappa_{1,t}^{1-\phi_1} \quad \text{(AT3.20)}$$

$$x_{2,t} = u_{2,t}\Omega_2\kappa_{2,t}^{1-\phi_2} \quad \text{(AT3.21)}$$

$$z_t = u_{z,t}\Omega_z\kappa_{z,t}^{1-\psi} \quad \text{(AT3.22)}$$

$$1 = u_{1,t} + u_{2,t} + u_{z,t} \quad \text{(AT3.23)}$$

Notes The endogenous variables are $K_{t+1}/\bar{h}_{t+1}$, γ_{t+1}, $\bar{e}_t$, $\pi^m_{1,t}/\bar{h}_t$, l_t, r_t, q_t, r^k_t, w_t, $x_{1,t} \equiv X_{1,t}/H_t$, $x_{2,t} \equiv X_{2,t}/H_t$, $z_t \equiv Z_t/H_t$, $u_{1,t} \equiv H_{1,t}/H_t$, $u_{2,t} \equiv H_{2,t}/H_t$, $u_{z,t} \equiv H_{z,t}/H_t$, $\kappa_t \equiv K_t/H_t$, $\kappa_{1,t} \equiv K_{1,t}/H_{1,t}$, $\kappa_{2,t} \equiv K_{2,t}/H_{2,t}$, $\kappa_{z,t} \equiv K_{z,t}/H_{z,t}$, $mc^x_{1,t}$, Ξ_t, p_t, and $y_t \equiv Y_t/H_t$. Of these, only $K_t/\bar{h}_t$ is predetermined at time t.

Table A.4: Features of the steady-state growth path (TO case)

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
θ	0.2125	0.2125	0.3000	0.2125	0.2125	0.2125	0.2125	0.2125	0.2125
ϕ_e	5.2998	5.2998	5.2998	6.0000	5.2998	5.2998	5.2998	5.2998	5.2998
ε		0.0800	0.0800	0.0800	0.1600	0.0800	0.0800	0.0800	0.8000
σ	2.0000	2.0000	2.0000	2.0000	2.0000	4.0000	2.0000	2.0000	2.0000
α	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.7000	0.5000	0.5000
ϕ (or ϕ_1)	0.8000	0.8000	0.8000	0.8000	0.8000	0.8000	0.8000	0.6000	0.8000
ψ	0.3708	0.3708	0.3708	0.3708	0.3708	0.3708	0.3708	0.3708	0.8000
y^*	1.0000	1.1276	1.0057	1.0651	1.1260	1.0512	1.4558	1.0948	1.3839
x_1^*	0.4417	0.1263	0.1126	0.1193	0.1262	0.1637	0.2416	0.0519	0.1559
x_2^*	0.4417	0.7364	0.6562	0.6955	0.7358	0.7120	0.5557	0.7925	0.9087
i^*	0.1165	0.0794	0.0560	0.0666	0.0786	0.0995	0.0337	0.0925	0.2417
l^*	0.1000	0.0727	0.0777	0.0805	0.0720	0.0880	0.0317	0.0750	0.0867
e^*		0.0112	0.0112	0.0111	0.0222	0.0050	0.0282	0.0072	0.0118
γ^*	1.0976	0.8541	1.2758	1.0475	0.8476	0.9925	0.4439	0.8750	0.9811
$\gamma_a^* \times 100\%$	2.5000	2.0792	2.7790	2.4175	2.0674	2.3246	1.2319	2.1175	2.3050
$\gamma_{ca}^* \times 100\%$	2.5000		3.1607	2.8531				2.0692	2.8427
w^*	0.7500	0.7222	0.6439	0.6821	0.7213	0.7387	0.6546	0.7248	0.9307
$(r^k)^*$	4.1657	4.8460	7.6697	6.0896	4.8682	4.4264	7.1793	4.7757	1.7567
r^*	3.3219	3.6245	4.6843	4.1236	3.6340	3.4408	4.5175	3.5944	3.4547
$r_a^* \times 100\%$	5.0000	5.2371	5.9634	5.5972	5.2443	5.0950	5.8583	5.2142	5.1060
p^*	1.0000	2.4142	2.4142	2.4142	2.4142	1.4440	3.5386	3.9087	2.4142
$(mc^x)^*$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.7327	1.0000
q^*	1.0000	1.0845	1.3874	1.2259	1.0872	1.0331	1.3391	1.0761	0.4087
u_1^*	0.4712	0.1400	0.1399	0.1400	0.1400	0.1773	0.2953	0.0744	0.1340
u_2^*	0.4712	0.8158	0.8154	0.8157	0.8161	0.7711	0.6791	0.8747	0.7811
u_z^*	0.0576	0.0442	0.0448	0.0444	0.0439	0.0516	0.0256	0.0509	0.0849
κ^*	0.0600	0.0468	0.0264	0.0352	0.0465	0.0542	0.0262	0.0538	0.1325
κ_1^*	0.0450	0.0373	0.0210	0.0280	0.0370	0.0417	0.0228	0.1012	0.1325
κ_2^*	0.0450	0.0373	0.0210	0.0280	0.0370	0.0417	0.0228	0.0379	0.1325
κ_z^*	0.3055	0.2528	0.1424	0.1900	0.2614	0.2831	0.1547	0.2575	0.1325
ζ^*	0.0840	0.0663	0.0373	0.0496	0.0653	0.0762	0.0377	0.0763	0.1856
$(\pi_{1,t}^m/\bar{h}_t)^*$	0.0000	0.2530	0.2245	0.2377	0.2510	0.1023	0.8833	0.1600	0.3090

Notes The perfectly competitive steady-state equilibrium (without rent seeking) is reported in column (a). Column (b) reports on the benchmark rent-seeking equilibrium. Columns (c)–(i) report on some alternative rent-seeking equilibria for different values of, respectively, θ, ϕ_e, ε, σ, α, ϕ_1, and ψ.

A.4 Education in the rent share function

- An individual's education level features in the share function.
- We change the share function to:

$$s_t(\eta) \equiv \frac{\eta \left[l_t(\eta)^{\xi} e_t(\eta)\right]^{\varepsilon}}{E_t}, \qquad \xi > 0,$$

where E_t is given by:

$$E_t \equiv \int_{\eta_L}^{\eta_H} \eta \left[l_t(\eta)^{\xi} e_t(\eta)\right]^{\varepsilon} dF(\eta).$$

- Note that:

$$\frac{\partial s_t(\eta)}{\partial l_t(\eta)} = \frac{\varepsilon \xi \eta \left[l_t(\eta)^{\xi} e_t(\eta)\right]^{\varepsilon}}{l_t(\eta) E_t} = \frac{\varepsilon \xi s_t(\eta)}{l_t(\eta)}, \qquad \frac{\partial s_t(\eta)}{\partial e_t(\eta)} = \frac{\varepsilon \eta \left[l_t(\eta)^{\xi} e_t(\eta)\right]^{\varepsilon}}{e_t(\eta) E_t} = \frac{\varepsilon s_t(\eta)}{e_t(\eta)}.$$

- Rent-seeking revenues accrue to the young (base case).
- Lagrangian:

$$\mathcal{L}_t^y \equiv \ln c_t^y(\eta) + \beta \ln c_{t+1}^o(\eta) + \mu_t \Bigg[s_t(\eta) \Pi_{1,t}^m + W_t \bar{h}_t \left[1 - e_t(\eta) - l_t(\eta)\right] + \frac{\lambda W_{t+1}}{1 + R_{t+1}^n} \bar{h}_t \left[1 + \phi_e \frac{l_t(\eta)^{1-\theta}}{1-\theta}\right] - P_{V,t} c_t^y(\eta) - \frac{P_{V,t+1} c_{t+1}^o(\eta)}{1 + R_{t+1}^n} \Bigg].$$

- First-order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}_t^y}{\partial c_t^y(\eta)} &= \frac{1}{c_t^y(\eta)} - \mu_t P_{V,t} = 0, \\
\frac{\partial \mathcal{L}_t^y}{\partial c_{t+1}^o(\eta)} &= \frac{\beta}{c_{t+1}^o(\eta)} - \frac{\mu_t P_{V,t+1}}{1 + R_{t+1}^n} = 0, \\
\frac{\partial \mathcal{L}_t^y}{\partial l_t(\eta)} &= \mu_t \left[\frac{\Pi_{1,t}^m}{\bar{h}_t} \frac{\partial s_t(\eta)}{\partial l_t(\eta)} - W_t + \frac{W_{t+1}}{1 + R_{t+1}^n} \lambda \phi_e l_t(\eta)^{-\theta} \right] \bar{h}_t = 0, \\
\frac{\partial \mathcal{L}_t^y}{\partial e_t(\eta)} &= \mu_t \left[\frac{\Pi_{1,t}^m}{\bar{h}_t} \frac{\partial s_t(\eta)}{\partial e_t(\eta)} - W_t \right] \bar{h}_t = 0.
\end{aligned}$$

- Substituting the first two expressions into the budget constraint (A.16) we find:

$$\begin{aligned}
P_{V,t} c_t^y(\eta) &= \frac{1}{1+\beta} HW_t^y(\eta), \\
\frac{P_{V,t+1} c_{t+1}^o(\eta)}{1 + R_{t+1}^n} &= \frac{\beta}{1+\beta} HW_t^y(\eta).
\end{aligned}$$

- Using the share function in the final two expressions we obtain:

$$W_t = \frac{\Pi^m_{1,t}}{\bar{h}_t}\frac{\varepsilon\xi\eta\left[l_t(\eta)^{\xi}e_t(\eta)\right]^{\varepsilon}}{l_t(\eta)E_t} + \frac{W_{t+1}}{1+R^n_{t+1}}\lambda\phi_e l_t(\eta)^{-\theta},$$
$$W_t = \frac{\Pi^m_{1,t}}{\bar{h}_t}\frac{\varepsilon\eta\left[l_t(\eta)^{\xi}e_t(\eta)\right]^{\varepsilon}}{e_t(\eta)E_t}.$$

- Combining we find:

$$e_t(\eta) = \left(\frac{\varepsilon\eta\Pi^m_{1,t}}{W_t\bar{h}_tE_t}\right)^{1/(1-\varepsilon)} l_t(\eta)^{\varepsilon\xi/(1-\varepsilon)},$$
$$1 = \xi\left(\frac{\varepsilon\eta\Pi^m_{1,t}}{W_t\bar{h}_tE_t}\right)^{1/(1-\varepsilon)} l_t(\eta)^{[\varepsilon(1+\xi)-1]/(1-\varepsilon)} + \frac{W_{t+1}}{(1+R^n_{t+1})W_t}\lambda\phi_e l_t(\eta)^{-\theta}.$$

- We also find:

$$E_t \equiv \int_{\eta_L}^{\eta_H} \eta l_t(\eta)^{\varepsilon\xi}e_t(\eta)^{\varepsilon}dF(\eta),$$
$$= \int_{\eta_L}^{\eta_H} \eta l_t(\eta)^{\varepsilon\xi}\left[\left(\frac{\varepsilon\eta\Pi^m_{1,t}}{W_t\bar{h}_tE_t}\right)^{1/(1-\varepsilon)} l_t(\eta)^{\varepsilon\xi/(1-\varepsilon)}\right]^{\varepsilon} dF(\eta),$$
$$E_t^{1/(1-\varepsilon)} = \left(\frac{\varepsilon\Pi^m_{1,t}}{W_t\bar{h}_t}\right)^{\varepsilon/(1-\varepsilon)} \int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} l_t(\eta)^{\varepsilon\xi+\varepsilon^2\xi/(1-\varepsilon)}dF(\eta),$$
$$E_t = \left(\frac{\varepsilon\Pi^m_{1,t}}{W_t\bar{h}_t}\right)^{\varepsilon}\left[\int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} l_t(\eta)^{\varepsilon\xi/(1-\varepsilon)}dF(\eta)\right]^{1-\varepsilon}.$$

- We thus find that:

$$e_t(\eta) = \frac{\varepsilon\Pi^m_{1,t}}{W_t\bar{h}_t}\frac{\eta^{1/(1-\varepsilon)}l_t(\eta)^{\varepsilon\xi/(1-\varepsilon)}}{\int_{\eta_L}^{\eta_H}\eta^{1/(1-\varepsilon)}l_t(\eta)^{\varepsilon\xi/(1-\varepsilon)}dF(\eta)}.$$

- It easily follows that total (and average) wasted labour is:

$$\bar{e}_t \equiv \int_{\eta_L}^{\eta_H} e_t(\eta)dF(\eta) = \frac{\varepsilon\Pi^m_{1,t}}{W_t\bar{h}_t}.$$

- To find $l_t(\eta)$ we need to be able to solve:

$$1 = E_t^{-1/(1-\varepsilon)}\xi\left(\frac{\varepsilon\eta\Pi^m_{1,t}}{W_t\bar{h}_t}\right)^{1/(1-\varepsilon)} l_t(\eta)^{[\varepsilon(1+\xi)-1]/(1-\varepsilon)} + \frac{W_{t+1}}{(1+R^n_{t+1})W_t}\lambda\phi_e l_t(\eta)^{-\theta},$$
$$= \xi\left(\frac{\varepsilon\Pi^m_{1,t}}{W_t\bar{h}_t}\right)^{-\varepsilon/(1-\varepsilon)}\left[\int_{\eta_L}^{\eta_H}\eta^{1/(1-\varepsilon)}l_t(\eta)^{\varepsilon\xi/(1-\varepsilon)}dF(\eta)\right]^{-1}\left(\frac{\varepsilon\eta\Pi^m_{1,t}}{W_t\bar{h}_t}\right)^{1/(1-\varepsilon)} l_t(\eta)^{[\varepsilon(1+\xi)-1]/(1-\varepsilon)}$$
$$+ \frac{W_{t+1}}{(1+R^n_{t+1})W_t}\lambda\phi_e l_t(\eta)^{-\theta},$$

$$= \frac{\varepsilon\xi\Pi_{1,t}^{m}}{W_t\bar{h}_t}\frac{\eta^{1/(1-\varepsilon)}l_t(\eta)^{[\varepsilon(1+\xi)-1]/(1-\varepsilon)}}{\int_{\eta_L}^{\eta_H}\eta^{1/(1-\varepsilon)}l_t(\eta)^{\varepsilon\xi/(1-\varepsilon)}dF(\eta)} + \frac{W_{t+1}}{(1+R_{t+1}^{n})W_t}\lambda\phi_e l_t(\eta)^{-\theta}.$$

- Rewrite this expression to:

$$l_t(\eta) = \frac{\varepsilon\xi\Pi_{1,t}^{m}}{W_t\bar{h}_t}\frac{\eta^{1/(1-\varepsilon)}l_t(\eta)^{\varepsilon\xi/(1-\varepsilon)}}{\int_{\eta_L}^{\eta_H}\eta^{1/(1-\varepsilon)}l_t(\eta)^{\varepsilon\xi/(1-\varepsilon)}dF(\eta)} + \frac{W_{t+1}}{(1+R_{t+1}^{n})W_t}\lambda\phi_e l_t(\eta)^{1-\theta}.$$

- Total (and average) education time is thus:

$$\begin{aligned}
\bar{l}_t &\equiv \int_{\eta_L}^{\eta_H} l_t(\eta)dF(\eta), \\
&= \frac{\varepsilon\xi\Pi_{1,t}^{m}}{W_t\bar{h}_t} + \frac{W_{t+1}}{(1+R_{t+1}^{n})W_t}\lambda(1-\theta)\phi_e\frac{\int_{\eta_L}^{\eta_H} l_t(\eta)^{1-\theta}dF(\eta)}{1-\theta}, \\
&= \xi\bar{e}_t + \frac{\lambda(1-\theta)W_{t+1}}{(1+R_{t+1}^{n})W_t}\left[1+\gamma_{t+1}\right].
\end{aligned}$$

- For given values of the macro variables we find that learning time depends on innate rent-seeking aptitude. Complications:
  - Individual and aggregate growth rates differ:

$$\begin{aligned}
\gamma_{t+1}(\eta) &\equiv \frac{h_{t+1}^{o}(\eta) - h_t^{y}(\eta)}{h_t^{y}(\eta)} = \phi_e\frac{l_t(\eta)^{1-\theta}}{1-\theta}, \\
\gamma_{t+1} &\equiv \frac{H_{t+1}-H_t}{H_t} = \phi_e\frac{\int_{\eta_L}^{\eta_H} l_t(\eta)^{1-\theta}dF(\eta)}{1-\theta}.
\end{aligned}$$

  - Hard to compute even numerical solutions.
  - Use discretized uniform distribution with N equally likely values η_i in $[\eta_L, \eta_H]$. To allow for generalization to other discretized distributions we write the frequencies as s_i (see below for details).

- Human capital stock:

$$\begin{aligned}
H_t &\equiv \int_{\eta_L}^{\eta_H}\left[\lambda h_t^{o}(\eta) + \left[1-e_t(\eta)-l_t(\eta)\right]h_t^{y}(\eta)\right]dF(\eta), \\
h_t^{o}(\eta) &= \bar{h}_{t-1}\left[1+\phi_e\frac{l_{t-1}(\eta)^{1-\theta}}{1-\theta}\right], \\
h_t^{y}(\eta) &= \bar{h}_t,
\end{aligned}$$

or:

$$H_t = \int_{\eta_L}^{\eta_H}\left[\lambda\bar{h}_{t-1}\left[1+\phi_e\frac{l_{t-1}(\eta)^{1-\theta}}{1-\theta}\right] + \left[1-e_t(\eta)-l_t(\eta)\right]\bar{h}_t\right]dF(\eta),$$

$$
\begin{aligned}
&= \lambda \bar{h}_{t-1}\left[1+\gamma_t\right] + \bar{h}_t \int_{\eta_L}^{\eta_H} \left[1 - e_t(\eta) - l_t(\eta)\right] dF(\eta), \\
&= \bar{h}_t \left[1 + \lambda - \int_{\eta_L}^{\eta_H} \left[e_t(\eta) + l_t(\eta)\right] dF(\eta)\right], \\
&= \bar{h}_t \left[1 + \lambda - \bar{e}_t - \bar{l}_t\right].
\end{aligned}
$$

where we have used the fact that $\bar{h}_{t-1}\left[1+\gamma_t\right] = \bar{h}_t$.

- Human wealth at birth:

$$
\begin{aligned}
HW_t^y &\equiv \int_{\eta_L}^{\eta_H} HW_t^y(\eta)\, dF(\eta), \\
&= \int_{\eta_L}^{\eta_H} \left[W_t h_t^y(\eta)\left[1 - e_t(\eta) - l_t(\eta)\right] + \frac{\lambda W_{t+1} h_{t+1}^o(\eta)}{1 + R_{t+1}^n} + s_t(\eta)\Pi_{1,t}^m\right] dF(\eta), \\
\frac{HW_t^y}{\bar{h}_t} &= W_t + \frac{\Pi_{1,t}^m}{\bar{h}_t} - W_t \int_{\eta_L}^{\eta_H} \left[e_t(\eta) + l_t(\eta)\right] dF(\eta) + \frac{\lambda W_{t+1}(1+\gamma_{t+1})}{1 + R_{t+1}^n}, \\
&= W_t\left[1 - L_t^e\right] + (1-\varepsilon)\frac{\Pi_{1,t}^m}{\bar{h}_t} + \frac{\lambda W_{t+1}(1+\gamma_{t+1})}{1 + R_{t+1}^n}.
\end{aligned}
$$

- Income of the old generation:

$$
\begin{aligned}
I_t^o &\equiv \int_{\eta_L}^{\eta_H} I_t^o(\eta) dF(\eta) = \int_{\eta_L}^{\eta_H} \left[\lambda W_t h_t^o(\eta) + \left[(1-\delta)Q_t + R_t^k\right]\left[z_{t-1}^y(\eta) + k_{t-1}^y(\eta)\right]\right] dF(\eta), \\
&= \lambda W_t \int_{\eta_L}^{\eta_H} h_t^o(\eta) dF(\eta) + \left[(1-\delta)Q_t + R_t^k\right]\left[Z_{t-1} + (1-\delta)K_{t-1}\right], \\
\frac{I_t^o}{\bar{h}_t} &= \lambda W_t + \left[(1-\delta)Q_t + R_t^k\right]\frac{K_t}{\bar{h}_t}.
\end{aligned}
$$

- Demand for good 1:

$$
\begin{aligned}
\frac{X_{1,t}}{\bar{h}_t} = \frac{\alpha^\sigma p_{1,t}^{-\sigma}}{\alpha^\sigma p_{1,t}^{1-\sigma} + (1-\alpha)^\sigma} &\left[\frac{1}{1+\beta}\left(w_t\left[1 - \bar{l}_t\right] + (1-\varepsilon)\frac{\Pi_{1,t}^m}{\bar{h}_t} + \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1 + r_{t+1}}\right)\right. \\
&\left. + \lambda w_t + \left[(1-\delta)q_t + r_t^k\right]\frac{K_t}{\bar{h}_t}\right].
\end{aligned}
$$

- Demand for good 2:

$$
\begin{aligned}
\frac{X_{2,t}}{\bar{h}_t} = \frac{(1-\alpha)^\sigma}{\alpha^\sigma p_{1,t}^{1-\sigma} + (1-\alpha)^\sigma} &\left[\frac{1}{1+\beta}\left(w_t\left[1 - \bar{l}_t\right] + (1-\varepsilon)\frac{\Pi_{1,t}^m}{\bar{h}_t} + \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1 + r_{t+1}}\right)\right. \\
&\left. + \lambda w_t + \left[(1-\delta)q_t + r_t^k\right]\frac{K_t}{\bar{h}_t}\right].
\end{aligned}
$$

- The demand for new investment goods follows from:

$$q_t \left[\frac{Z_t + (1-\delta) K_t}{\bar{h}_t} \right] = \frac{\Pi_{1,t}^m}{\bar{h}_t} + w_t \int_{\eta_L}^{\eta_H} [1 - e_t(\eta) - l_t(\eta)] \, dF(\eta) - \frac{1}{1+\beta} \frac{HW_t}{\bar{h}_t}$$
$$= \frac{\beta}{1+\beta} \left[w_t \left[1 - \bar{l}_t \right] + (1-\varepsilon) \frac{\Pi_{1,t}^m}{\bar{h}_t} \right] - \frac{1}{1+\beta} \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1+r_{t+1}}.$$

After simplifying we obtain:

$$q_t \frac{Z_t}{\bar{h}_t} = \frac{\beta}{1+\beta} \left[w_t \left[1 - \bar{l}_t \right] + (1-\varepsilon) \frac{\Pi_{1,t}^m}{\bar{h}_t} \right]$$
$$- \frac{1}{1+\beta} \frac{\lambda w_{t+1}}{1+r_{t+1}} \left[1 + \gamma_{t+1} \right] - q_t (1-\delta) \frac{K_t}{\bar{h}_t}.$$

- Resulting model can be found in Table A.5
- Features of the steady-state growth path: Table A.6
- For the visualizations we use the uniform distribution for η:
  - Density and distribution functions:

$$f(\eta) \equiv \frac{1}{\eta_H - \eta_L}, \qquad F(\eta) \equiv \frac{\eta - \eta_L}{\eta_H - \eta_L} \quad (\text{for } \eta_L \leq \eta \leq \eta_H).$$

  - Weight:

$$\int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta) = \frac{1-\varepsilon}{(2-\varepsilon)(\eta_H - \eta_L)} \left[\eta_H^{(2-\varepsilon)/(1-\varepsilon)} - \eta_L^{(2-\varepsilon)/(1-\varepsilon)} \right].$$

  - Rent-seeking time:

$$e_t(\eta) = \frac{\varepsilon \Pi_{1,t}^m}{W_t \bar{h}_t} \frac{(2-\varepsilon)(\eta_H - \eta_L)}{1-\varepsilon} \frac{\eta^{1/(1-\varepsilon)}}{\eta_H^{(2-\varepsilon)/(1-\varepsilon)} - \eta_L^{(2-\varepsilon)/(1-\varepsilon)}}.$$

  - Total rent-seeking effort:

$$E_t = \left[\frac{\varepsilon \Pi_{1,t}^m}{W_t \bar{h}_t} \right]^{\varepsilon} \left[\frac{1-\varepsilon}{(2-\varepsilon)(\eta_H - \eta_L)} \left[\eta_H^{(2-\varepsilon)/(1-\varepsilon)} - \eta_L^{(2-\varepsilon)/(1-\varepsilon)} \right] \right]^{1-\varepsilon}.$$

  - Share function:

$$s_t(\eta) = \frac{(2-\varepsilon)(\eta_H - \eta_L)}{1-\varepsilon} \frac{\eta^{1/(1-\varepsilon)}}{\eta_H^{(2-\varepsilon)/(1-\varepsilon)} - \eta_L^{(2-\varepsilon)/(1-\varepsilon)}}.$$

  - Hence:

$$e_t(\eta) = \frac{\varepsilon \Pi_{1,t}^m}{W_t \bar{h}_t} s_t(\eta).$$

- Numerical issues:

  - Equations (AT5.4)–(AT5.6) are somewhat complicated.
  - By discretizing the distribution for η, however, we can easily rewrite the model in tractable terms.
  - Example: uniform distribution between η_L and η_U can be approximated with N equally spaced discrete values of η. With N intervals we place the η value for each interval in the middle.
  - The frequency of η_i is denoted by $s_i = 1/N$, such that:

$$\sum_{i=1}^{N} s_i = 1,$$

$$\eta_1 \equiv \eta_L + \frac{\eta_H - \eta_L}{2N},$$

$$\eta_{i+1} = \eta_i + \frac{\eta_H - \eta_L}{N}, \qquad \text{for } i = 1, 2, \ldots, N-1.$$

    Note that we can also write:

$$\eta_i \equiv \eta_L + \frac{2i-1}{2N}(\eta_H - \eta_L), \qquad \text{for } i = 1, 2, \ldots, N.$$

  - Equations (AT5.4)–(AT5.6) can now be written as:

$$\gamma_{t+1} = \phi_e \frac{\sum_{i=1}^{N} s_i l_t(\eta_i)^{1-\theta}}{1-\theta}, \tag{AT5.4*}$$

$$l_t(\eta_i) = \xi \bar{e}_t \frac{\eta_i^{1/(1-\varepsilon)} l_t(\eta_i)^{\varepsilon\xi/(1-\varepsilon)}}{\sum_{i=1}^{N} s_i \eta_i^{1/(1-\varepsilon)} l_t(\eta_i)^{\varepsilon\xi/(1-\varepsilon)}}, + \frac{w_{t+1}}{(1+r_{t+1})w_t} \lambda \phi_e l_t(\eta_i)^{1-\theta}, \qquad i = 1, 2, \ldots, N, \tag{AT4.5*}$$

$$\bar{l}_t = \sum_{i=1}^{N} s_i l_t(\eta_i). \tag{AT4.6*}$$

    This is a nonlinear system of $N+2$ equations in $N+2$ variables: γ_{t+1}, $\bar{l}_t$, and $l_t(\eta_i)$ for $i = 1, 2, \ldots, N$.

  - Other (more complicated) distributions can be discretized and used in a similar way.

Table A.5: Rent-seeking and growth in the three-sector model (education scenario)

$$(1+\gamma_{t+1})q_t\frac{K_{t+1}}{\bar{h}_{t+1}} = \frac{1}{1+\beta}\left[\beta(1-\varepsilon)\frac{\pi^m_{1,t}}{\bar{h}_t} + \beta w_t\left(1-\bar{l}_t\right) - \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1+r_{t+1}}\right] \tag{AT5.1}$$

$$\begin{aligned}\frac{\pi^m_{1,t}}{\bar{h}_t} &= \frac{\Xi_t}{1+\beta-(1-\varepsilon)\Xi_t}\left[w_t\left(1-\bar{l}_t\right) + \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1+r_{t+1}}\right] \\ &\quad + \frac{(1+\beta)\Xi_t}{1+\beta-(1-\varepsilon)\Xi_t}\left[\lambda w_t + \left((1-\delta)\,q_t + r^k_t\right)\frac{K_t}{\bar{h}_t}\right]\end{aligned} \tag{AT5.2}$$

$$w_t\bar{e}_t = \varepsilon\frac{\pi^m_{1,t}}{\bar{h}_t} \tag{AT5.3}$$

$$\gamma_{t+1} = \phi_e\frac{\int_{\eta_L}^{\eta_H} l_t(\eta)^{1-\theta}dF(\eta)}{1-\theta} \tag{AT5.4}$$

$$l_t(\eta) = \xi\bar{e}_t\frac{\eta^{1/(1-\varepsilon)}l_t(\eta)^{\varepsilon\xi/(1-\varepsilon)}}{\int_{\eta_L}^{\eta_H}\eta^{1/(1-\varepsilon)}l_t(\eta)^{\varepsilon\xi/(1-\varepsilon)}dF(\eta)} + \frac{w_{t+1}}{(1+r_{t+1})w_t}\lambda\phi_e l_t(\eta)^{1-\theta} \tag{AT5.5}$$

$$\bar{l}_t = \int_{\eta_L}^{\eta_H} l_t(\eta)dF(\eta) \tag{AT5.6}$$

$$1+r_{t+1} \equiv \frac{r^k_{t+1}+(1-\delta)q_{t+1}}{q_t} \tag{AT5.7}$$

$$w_t = \phi_1 mc^x_{1,t}\Omega_1\kappa_{1,t}^{1-\phi_1} = \phi_2\Omega_2\kappa_{2,t}^{1-\phi_2} = \psi q_t\Omega_z\kappa_{z,t}^{1-\psi} \tag{AT5.8)-(AT5.10}$$

$$r^k_t = (1-\phi_1)mc^x_{1,t}\Omega_1\kappa_{1,t}^{-\phi_1} = (1-\phi_2)\Omega_2\kappa_{2,t}^{-\phi_2} = (1-\psi)q_t\Omega_z\kappa_{z,t}^{-\psi} \tag{AT5.11)-(AT5.13}$$

$$\kappa_t = u_{1,t}\kappa_{1,t} + u_{2,t}\kappa_{2,t} + u_{z,t}\kappa_{z,t} \tag{AT5.14}$$

$$z_t = \left(\frac{1+\lambda-\bar{e}_{t+1}-\bar{l}_{t+1}}{1+\lambda-\bar{e}_t-\bar{l}_t}\right)(1+\gamma_{t+1})\kappa_{t+1} - (1-\delta)\kappa_t \tag{AT5.15}$$

$$\kappa_t = \frac{1}{1+\lambda-\bar{e}_t-\bar{l}_t}\frac{K_t}{\bar{h}_t} \tag{AT5.16}$$

$$y_t = p_t x_{1,t} + x_{2,t} + q_t z_t \tag{AT5.17}$$

$$\Xi_t \equiv \frac{\alpha^\sigma p_t^{1-\sigma}}{\alpha^\sigma p_t^{1-\sigma} + \sigma(1-\alpha)^\sigma} \tag{AT5.18}$$

$$p_t = \frac{\alpha^\sigma p_t^{1-\sigma} + \sigma(1-\alpha)^\sigma}{(\sigma-1)(1-\alpha)^\sigma}mc^x_{1,t} \tag{AT5.19}$$

Table A.5: Continued

$$p_{1,t}x_{1,t} = \frac{\alpha^\sigma p_t^{1-\sigma}}{\alpha^\sigma p_t^{1-\sigma} + (1-\alpha)^\sigma}\frac{1}{1+\lambda-\bar{e}_t-\bar{l}_t}\left[\frac{1}{1+\beta}\left((1-\varepsilon)\frac{\pi^m_{1,t}}{\bar{h}_t} + w_t\left(1-\bar{l}_t\right) + \frac{\lambda w_{t+1}(1+\gamma_{t+1})}{1+r_{t+1}}\right)\right.$$
$$\left. + \lambda w_t + \left((1-\delta)\,q_t + r^k_t\right)\frac{K_t}{\bar{h}_t}\right] \tag{AT5.20}$$

$$x_{1,t} = u_{1,t}\Omega_1\kappa_{1,t}^{1-\phi_1} \tag{AT5.21}$$

$$x_{2,t} = u_{2,t}\Omega_2\kappa_{2,t}^{1-\phi_2} \tag{AT5.22}$$

$$z_t = u_{z,t}\Omega_z\kappa_{z,t}^{1-\psi} \tag{AT5.23}$$

$$1 = u_{1,t} + u_{2,t} + u_{z,t} \tag{AT5.24}$$

Notes The endogenous variables are $K_{t+1}/\bar{h}_{t+1}$, γ_{t+1}, $\bar{e}_t$, $\pi^m_{1,t}/\bar{h}_t$, $l_t(\eta)$, $\bar{l}_t$, r_t, q_t, r^k_t, w_t, $x_{1,t} \equiv X_{1,t}/H_t$, $x_{2,t} \equiv X_{2,t}/H_t$, $z_t \equiv Z_t/H_t$, $u_{1,t} \equiv H_{1,t}/H_t$, $u_{2,t} \equiv H_{2,t}/H_t$, $u_{z,t} \equiv H_{z,t}/H_t$, $\kappa_t \equiv K_t/H_t$, $\kappa_{1,t} \equiv K_{1,t}/H_{1,t}$, $\kappa_{2,t} \equiv K_{2,t}/H_{2,t}$, $\kappa_{z,t} \equiv K_{z,t}/H_{z,t}$, $mc^x_{1,t}$, Ξ_t, p_t, and $y_t \equiv Y_t/H_t$. Of these, only $K_t/\bar{h}_t$ is predetermined at time t.

Table A.6: Features of the steady-state growth path (education case)

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
θ	0.2125	0.2125	0.3000	0.2125	0.2125	0.2125	0.2125	0.2125	0.2125
ϕ_e	5.2998	5.2998	5.2998	6.0000	5.2998	5.2998	5.2998	5.2998	5.2998
ε	0.0800	0.0800	0.0800	0.0800	0.1600	0.0800	0.0800	0.0800	0.0800
σ	2.0000	2.0000	2.0000	2.0000	2.0000	4.0000	2.0000	2.0000	2.0000
α	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.7000	0.5000	0.5000
ϕ (or ϕ_1)	0.8000	0.8000	0.8000	0.8000	0.8000	0.8000	0.8000	0.6000	0.8000
ψ	0.3708	0.3708	0.3708	0.3708	0.3708	0.3708	0.3708	0.3708	0.8000
y^*	1.2516	1.1782	1.0748	1.1104	1.1219	1.0678	1.6791	1.1214	1.3970
x_1^*	0.1315	0.1241	0.1130	0.1170	0.1187	0.1616	0.2429	0.0513	0.1494
x_2^*	0.7666	0.7233	0.6588	0.6817	0.6918	0.7027	0.5586	0.7773	0.8708
i^*	0.1825	0.1490	0.1126	0.1236	0.1244	0.1281	0.2521	0.1356	0.4048
$\bar{l}^*$	0.1399	0.1716	0.1680	0.1797	0.1966	0.1311	0.3057	0.1384	0.2068
$\bar{e}^*$	0.0254	0.0249	0.0249	0.0247	0.0481	0.0105	0.0748	0.0164	0.0226
γ^*	1.4299	1.6673	2.1592	1.9583	1.8368	1.3548	2.6085	1.4101	1.9358
$\gamma_a^* \times 100\%$	3.0037	3.3243	3.9089	3.6816	3.5367	2.8961	4.3704	2.9757	3.6551
$\gamma_{ca}^* \times 100\%$	2.5000		3.1607	2.8531				2.0692	2.8427
w^*	0.7806	0.7355	0.6706	0.6932	0.7016	0.7403	0.7380	0.7285	0.9486
$(r^k)^*$	3.5501	4.5036	6.5193	5.7084	5.4386	4.3889	4.4434	4.6807	1.6281
r^*	3.0243	3.4753	4.2831	3.9770	3.8700	3.4240	3.4484	3.5532	3.1401
$r_a^* \times 100\%$	4.7506	5.1221	5.7052	5.4951	5.4187	5.0817	5.1010	5.1826	4.8497
p^*	2.4142	2.4142	2.4142	2.4142	2.4142	1.4440	3.5386	3.8911	2.4142
$(mc^x)^*$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.7240	1.0000
q^*	0.9178	1.0427	1.2716	1.1841	1.1538	1.0284	1.0352	1.0645	0.4087
u_1^*	0.1348	0.1350	0.1349	0.1350	0.1353	0.1746	0.2633	0.0729	0.1260
u_2^*	0.7856	0.7867	0.7860	0.7867	0.7888	0.7594	0.6055	0.8536	0.7344
u_z^*	0.0796	0.0783	0.0792	0.0783	0.0759	0.0660	0.1312	0.0735	0.1395
κ^*	0.0803	0.0593	0.0375	0.0441	0.0464	0.0583	0.0730	0.0602	0.1457
κ_1^*	0.0550	0.0408	0.0257	0.0304	0.0323	0.0422	0.0415	0.1038	0.1457
κ_2^*	0.0550	0.0408	0.0257	0.0304	0.0323	0.0422	0.0415	0.0389	0.1457
κ_z^*	0.3730	0.2771	0.1745	0.2060	0.2189	0.2862	0.2818	0.2640	0.1457
ζ^*	0.1072	0.0773	0.0490	0.0572	0.0583	0.0792	0.0818	0.0810	0.1851
$(\pi_{1,t}^m/\bar{h}_t)^*$	0.2482	0.2288	0.2089	0.2143	0.2107	0.0975	0.6902	0.1496	0.2685
$l^*(\eta_1)$	0.1399	0.0934	0.1065	0.1008	0.0673	0.0934	0.1106	0.0840	0.1304
$l^*(\eta_2)$	0.1399	0.1115	0.1202	0.1190	0.0929	0.1018	0.1572	0.0964	0.1477
$l^*(\eta_3)$	0.1399	0.1294	0.1341	0.1371	0.1204	0.1103	0.2017	0.1088	0.1651
$l^*(\eta_4)$	0.1399	0.1469	0.1479	0.1547	0.1490	0.1188	0.2449	0.1210	0.1823
$l^*(\eta_5)$	0.1399	0.1641	0.1616	0.1721	0.1784	0.1272	0.2872	0.1331	0.1992
$l^*(\eta_6)$	0.1399	0.1810	0.1751	0.1892	0.2085	0.1356	0.3290	0.1449	0.2159
$l^*(\eta_7)$	0.1399	0.1977	0.1886	0.2061	0.2394	0.1438	0.3703	0.1566	0.2324
$l^*(\eta_8)$	0.1399	0.2143	0.2020	0.2228	0.2709	0.1520	0.4114	0.1682	0.2487
$l^*(\eta_9)$	0.1399	0.2306	0.2153	0.2393	0.3031	0.1602	0.4522	0.1797	0.2649
$l^*(\eta_{10})$	0.1399	0.2468	0.2285	0.2557	0.3358	0.1682	0.4928	0.1911	0.2809

Notes The benchmark rent-seeking equilibrium is reported in column (a). Column (b) reports on the rent-seeking equilibrium with education-augmented rent-seeking. Columns (c)–(i) report on some alternative rent-seeking equilibria for different values of, respectively, θ, ϕ_e, ε, σ, α, ϕ_1, and ψ.

A.5 Physical capital externality

- Physical capital externality as the source of endogenous growth.
- No education decision: human capital is constant.
- To make the model compatible with the human-capital based growth model we assume that the time endowments are $\lambda^y = 0.9$ (instead of 1) and $\lambda^o = 0.5$ (as before).

A.5.1 Individual agents

- Utility function:

$$\Lambda_t^y(\eta) \equiv \ln c_t^y(\eta) + \beta \ln c_{t+1}^o(\eta). \tag{A.52}$$

  where $c_t^y(\eta)$ and $c_{t+1}^y(\eta)$ are defined as:

$$\begin{aligned} c_t^y(\eta) &\equiv \left[\alpha x_{1,t}^y(\eta)^{1-1/\sigma} + (1-\alpha) x_{2,t}^y(\eta)^{1-1/\sigma}\right]^{1/(1-1/\sigma)}, \\ c_{t+1}^o(\eta) &\equiv \left[\alpha x_{1,t+1}^o(\eta)^{1-1/\sigma} + (1-\alpha) x_{2,t+1}^o(\eta)^{1-1/\sigma}\right]^{1/(1-1/\sigma)}. \end{aligned}$$

- Budget constraint during youth:

$$P_{1,t} x_{1,t}^y(\eta) + P_{2,t} x_{2,t}^y(\eta) + Q_t \left[z_t^y(\eta) + k_t^y(\eta)\right] = I_t^y(\eta). \tag{A.53}$$

  where

$$I_t^y(\eta) = W_t \bar{h}_t \left[\lambda^y - e_t(\eta)\right] + s_t(\eta) \Pi_{1,t}^m. \tag{A.54}$$

  - W_t is the wage rate on standardized efficiency units of labour.
  - $e_t(\eta)$ is time spent lobbying.
  - $\bar{h}_t$ is the average human capital level in the economy at the start of time t (constant, can be normalized to $\bar{h}_t = 1$).

- Budget constraint during old-age:

$$P_{1,t+1} x_{1,t+1}^o(\eta) + P_{2,t+1} x_{2,t+1}^o(\eta) = I_{t+1}^o(\eta).$$

  with:

$$I_{t+1}^o(\eta) \equiv \lambda^o W_{t+1} \bar{h}_t + \left[(1-\delta) Q_{t+1} + R_{t+1}^k\right] \left[z_t^y(\eta) + k_t^y(\eta)\right].$$

A.5.2 Firms

- With a capital externality we must impose the same technology on all three sectors, i.e. $\phi_1 = \phi_2 = \psi = \phi$ from here on. The model with a sector-specific external effect and different technologies can be formulated but it is very fragile (even for the competitive case).

- Consumption good i is produced with physical and human capital:

$$X_{i,t} = \Omega_t H_{i,t}^{\phi} K_{i,t}^{1-\phi}.$$

  - diminishing returns to both factors, i.e. $0 < \phi < 1$.
  - both factors are perfectly mobile across sectors.
  - the productivity term is time-dependent and taken as given by individual firms (see below).

- Profit in sector i is:

$$\Pi_{i,t} = P_{i,t} X_{i,t} - W_t H_{i,t} - R_t^k K_{i,t}$$

  which gives:

$$R_t^k = (1-\phi) P_{i,t} \Omega_t H_{i,t}^{\phi} K_{i,t}^{-\phi},$$
$$W_t = \phi_i P_{i,t} \Omega_t H_{i,t}^{\phi-1} K_{i,t}^{1-\phi}.$$

- There exists an external effect on general productivity affecting all sectors equally:

$$\Omega_t = \Omega K_t^{\phi}, \tag{A.55}$$

  where K_t is the total stock of capital in the economy.

- Factor demands simplify to:

$$R_t^k = (1-\phi) P_{i,t} \Omega H_{i,t}^{\phi} \left(\frac{K_{i,t}}{K_t}\right)^{-\phi},$$
$$\frac{W_t}{K_t} = \phi P_{i,t} \Omega H_{i,t}^{\phi-1} \left(\frac{K_{i,t}}{K_t}\right)^{1-\phi}.$$

- If we use good X_2 (always produced competitively) as the numeraire commodity we find the competitive factor demands:

$$r_t^k = (1-\phi) p_t \Omega H_{1,t}^{\phi} \left(\frac{K_{1,t}}{K_t}\right)^{-\phi} = (1-\phi) \Omega H_{2,t}^{\phi} \left(\frac{K_{2,t}}{K_t}\right)^{-\phi},$$

$$\frac{w_t}{K_t} = \phi p_t \Omega H_{1,t}^{\phi-1}\left(\frac{K_{1,t}}{K_t}\right)^{1-\phi} = \phi \Omega H_{2,t}^{\phi-1}\left(\frac{K_{2,t}}{K_t}\right)^{1-\phi}.$$

with:

$$r_t^k \equiv \frac{R_t^k}{P_{2,t}}, \qquad w_t \equiv \frac{W_t}{P_{2,t}}, \qquad p_t \equiv \frac{P_{1,t}}{P_{2,t}}.$$

- Output in sector i is:

$$X_{i,t} = \Omega H_{i,t}^{\phi}\left(\frac{K_{i,t}}{K_t}\right)^{1-\phi} K_t.$$

- The total cost function is $TC_i^x(W_t, R_t^k, X_{i,t}) \equiv MC_i^x(W_t, R_t^k)X_{i,t}$ with:

$$MC_i^x(W_t/K_t, R_t^k) \equiv P_{2,t}\left(\frac{w_t}{\phi K_t}\right)^{\phi}\left(\frac{r_t^k}{1-\phi}\right)^{1-\phi}\frac{1}{\Omega} \qquad \Leftrightarrow$$
$$mc_i^x(w_t/K_t, r_t^k) = \left(\frac{w_t}{\phi K_t}\right)^{\phi}\left(\frac{r_t^k}{1-\phi}\right)^{1-\phi}\frac{1}{\Omega}.$$

- Since $P_{2,t} = MC_2^x(W_t/K_{2,t}, R_t^k)$ we find that $mc_2^x(w_t/K_t, r_t^k) = 1$, and thus, $mc_1^x(w_t/K_t, r_t^k) = 1$.

- The total stock of efficiency units of labour is:

$$H_t \equiv \bar{h}_t \int_{\eta_L}^{\eta_H} \left[\lambda^o + \lambda^y - e_t(\eta)\right] dF(\eta).$$

  - Units of 'old' and 'young' human capital are perfect substitutes.
  - Since $\bar{h}_t$ is constant we can set $\bar{h}_t = 1$ from here on.

- The investment good is also produced with units of physical and human capital:

$$Z_t = \Omega_t H_{z,t}^{\phi} K_{z,t}^{1-\phi}.$$

- The firm hires these inputs (from their owners) to maximize profit:

$$\Pi_t^z \equiv Q_t Z_t - W_t H_{z,t} - R_t^k K_{z,t},$$

which gives:

$$R_t^k = (1-\phi) Q_t \Omega_t H_{z,t}^{\phi} K_{z,t}^{-\phi},$$
$$W_t = \phi Q_t \Omega_t H_{z,t}^{\phi-1} K_{z,t}^{1-\phi}.$$

- Using the expression for Ω_t in (A.55) we find:

$$r_t^k = (1-\phi) q_t \Omega H_{z,t}^{\phi}.$$
$$\frac{w_t}{K_t} = \phi q_t \Omega H_{z,t}^{\phi-1} \left(\frac{K_{z,t}}{K_t}\right)^{1-\phi},$$

where q_t is:

$$q_t \equiv \frac{Q_t}{P_{2,t}}.$$

- Output is:

$$Z_t = \Omega H_{z,t}^{\phi} \left(\frac{K_{z,t}}{K_t}\right)^{1-\phi} K_t.$$

- Obviously, $mc^z(w_t/K_t, r_t^k) = 1$ so that perfect competition in the investment goods sector yields:

$$q_t = mc^z(w_t/K_t, r_t^k) \equiv \left(\frac{w_t}{\phi K_t}\right)^{\phi} \left(\frac{r_t^k}{1-\phi}\right)^{1-\phi} \frac{1}{\Omega} = 1.$$

A.5.3 Loose ends

- Capital accumulation:

$$K_{t+1} = Z_t + (1-\delta) K_t \tag{A.56}$$

- Stock of human capital available for productive use:

$$H_t = \lambda^o + \lambda^y - \bar{e}_t, \tag{A.57}$$
$$\bar{e}_t \equiv \int_{\eta_L}^{\eta_H} e_t(\eta) dF(\eta), \tag{A.58}$$

where we recall that $\bar{h}_t = 1$.

- Equilibrium in the investment goods market:

$$Z_t = \int_{\eta_L}^{\eta_H} z_t^y(\eta) dF(\eta). \tag{A.59}$$

- Equilibrium in the market for used capital goods:

$$\int_{\eta_L}^{\eta_H} k_t^y(\eta) dF(\eta) = (1-\delta) K_t. \tag{A.60}$$

- Equilibrium condition in the physical capital rental market:

$$K_t = K_{1,t} + K_{2,t} + K_{z,t}.$$

- Equilibrium condition in the human capital rental market:

$$H_t = H_{1,t} + H_{2,t} + H_{z,t}.$$

A.5.4 Model solution

- We know that:

$$X_t^y(\eta) \equiv P_{1,t}x_{1,t}^y(\eta) + P_{2,t}x_{2,t}^y(\eta) = P_{V,t}c_t^y(\eta),$$
$$X_{t+1}^o(\eta) \equiv P_{1,t+1}x_{1,t+1}^o(\eta) + P_{2,t+1}x_{2,t+1}^o(\eta) = P_{V,t+1}c_{t+1}^o(\eta),$$

where $X_t^y(\eta)$ is full consumption and $P_{V,t}$ is the true price index:

$$P_{V,t} \equiv \left[\alpha^\sigma P_{1,t}^{1-\sigma} + (1-\alpha)^\sigma P_{2,t}^{1-\sigma}\right]^{1/(1-\sigma)}.$$

- Useful results from duality theory:
  - The expenditure functions are $E_t^y(\eta) \equiv P_{V,t}c_t^y(\eta)$ and $E_{t+1}^o(\eta) \equiv P_{V,t+1}c_{t+1}^o(\eta)$ so we can recover the *Hicksian* demands for the underlying goods in the usual fashion (Shephard's Lemma):

$$x_{i,t}^y(\eta) = \frac{\partial E_t^y(\eta)}{\partial P_{i,t}} = \frac{\partial P_{V,t}}{\partial P_{i,t}}c_t^y(\eta), \qquad x_{i,t+1}^o(\eta) = \frac{\partial E_{t+1}^o(\eta)}{\partial P_{i,t+1}} = \frac{\partial P_{V,t+1}}{\partial P_{i,t+1}}c_{t+1}^o(\eta).$$

  - The indirect (sub)utility functions are $V_t^y(\eta) \equiv X_t^y(\eta)/P_{V,t}$ and $V_{t+1}^o(\eta) \equiv X_{t+1}^o(\eta)/P_{V,t+1}$ and the Marshallian demands for the underlying goods in the usual fashion (Roy's Identity):

$$x_{i,t}^y(\eta) = -\frac{\partial V_t^y(\eta)/\partial P_{i,t}}{\partial V_t^y(\eta)/\partial X_t^y(\eta)} = \frac{\partial P_{V,t}}{\partial P_{i,t}}\frac{X_t^y(\eta)}{P_{V,t}}, \qquad x_{i,t+1}^o(\eta) = -\frac{\partial V_{t+1}^o(\eta)/\partial P_{i,t+1}}{\partial V_{t+1}^o(\eta)/\partial X_{t+1}^o(\eta)} = \frac{\partial P_{V,t+1}}{\partial P_{i,t+1}}\frac{X_{t+1}^o(\eta)}{P_{V,t+1}}.$$

- Budget constraints for young and old:

$$P_{V,t}c_t^y(\eta) + Q_t\left[z_t^y(\eta) + k_t^y(\eta)\right] = W_t\left[\lambda^y - e_t(\eta)\right] + s_t(\eta)\Pi_{1,t}^m,$$
$$P_{V,t+1}c_{t+1}^o(\eta) = \lambda^o W_{t+1} + \left[(1-\delta)Q_{t+1} + R_{t+1}^k\right]\left[z_t^y(\eta) + k_t^y(\eta)\right].$$

- Define the 'nominal' interest rate as:

$$1 + R_{t+1}^n \equiv \frac{(1-\delta)Q_{t+1} + R_{t+1}^k}{Q_t}.$$

- Solve the old-age budget constraint for $[z_t^y(\eta) + k_t^y(\eta)]$:

$$z_t^y(\eta) + k_t^y(\eta) = \frac{P_{V,t+1} c_{t+1}^o(\eta) - \lambda^o W_{t+1}}{(1 + R_{t+1}^n) Q_t}.$$

- Substitute into the youth budget constraint to get the consolidated budget constraint in nominal terms:

$$P_{V,t} c_t^y(\eta) + \frac{P_{V,t+1} c_{t+1}^o(\eta)}{1 + R_{t+1}^n} = HW_t^y(\eta). \tag{A.61}$$

where human wealth during youth is:

$$HW_t^y(\eta) \equiv W_t \left[\lambda^y - e_t(\eta)\right] + \frac{\lambda^o W_{t+1}}{1 + R_{t+1}^n} + s_t(\eta) \Pi_{1,t}^m.$$

- Since there is no uncertainty ($\pi_m = 1$ for all t) we can solve the optimization problem in one go. In particular, the agents chooses $c_t^y(\eta)$, $c_{t+1}^o(\eta)$, and $e_t(\eta)$ to maximize (A.52) subject to the budget constraint (A.61).

- Note: we continue to use 'nominal' terms (and use the numeraire right at the end).

- Lagrangian:

$$\mathcal{L}_t^y \equiv \ln c_t^y(\eta) + \beta \ln c_{t+1}^o(\eta) + \mu_t \Bigg[s_t(\eta) \Pi_{1,t}^m + W_t \left[\lambda^y - e_t(\eta)\right] + \frac{\lambda^o W_{t+1}}{1 + R_{t+1}^n} - P_{V,t} c_t^y(\eta) - \frac{P_{V,t+1} c_{t+1}^o(\eta)}{1 + R_{t+1}^n} \Bigg]$$

- First-order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}_t^y}{\partial c_t^y(\eta)} &= \frac{1}{c_t^y(\eta)} - \mu_t P_{V,t} = 0, \\
\frac{\partial \mathcal{L}_t^y}{\partial c_{t+1}^o(\eta)} &= \frac{\beta}{c_{t+1}^o(\eta)} - \frac{\mu_t P_{V,t+1}}{1 + R_{t+1}^n} = 0, \\
\frac{\partial \mathcal{L}_t^y}{\partial e_t(\eta)} &= \mu_t \left[\Pi_{1,t}^m \frac{\partial s_t(\eta)}{\partial e_t(\eta)} - W_t \right] = 0.
\end{aligned}$$

- Substituting the first two into the budget constraint (A.61) we find:

$$\begin{aligned}
P_{V,t} c_t^y(\eta) &= \frac{1}{1 + \beta} HW_t^y(\eta), \\
\frac{P_{V,t+1} c_{t+1}^o(\eta)}{1 + R_{t+1}^n} &= \frac{\beta}{1 + \beta} HW_t^y(\eta).
\end{aligned}$$

- For the success function $s_t(\eta) = \eta e_t(\eta)^\varepsilon / E_t$ we find:

$$\varepsilon \Pi_{1,t}^m \frac{\eta e_t(\eta)^{\varepsilon-1}}{E_t} = W_t \qquad \Leftrightarrow \qquad e_t(\eta) = \left[\frac{\varepsilon \Pi_{1,t}^m}{W_t} \frac{\eta}{E_t}\right]^{1/(1-\varepsilon)}.$$

- It follows that total rent-seeking effort E_t and wasted labour $\bar{e}_t$ amount to:

$$E_t \equiv \int_{\eta_L}^{\eta_H} \eta e_t(\eta)^\varepsilon dF(\eta) = \left[\frac{\varepsilon \Pi_{1,t}^m}{W_t E_t}\right]^{\varepsilon/(1-\varepsilon)} \int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta),$$
$$\bar{e}_t \equiv \int_{\eta_L}^{\eta_H} e_t(\eta) dF(\eta) = \left[\frac{\varepsilon \Pi_{1,t}^m}{W_t E_t}\right]^{1/(1-\varepsilon)} \int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta).$$

- Solving for E_t gives:

$$E_t = \left[\frac{\varepsilon \Pi_{1,t}^m}{W_t}\right]^\varepsilon \left[\int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta)\right]^{1-\varepsilon}.$$

- Solving for $e_t(\eta)$ gives:

$$e_t(\eta) = \frac{\varepsilon \Pi_{1,t}^m}{W_t} \frac{\eta^{1/(1-\varepsilon)}}{\int_{\eta_L}^{\eta_H} \eta^{1/(1-\varepsilon)} dF(\eta)}. \tag{A.62}$$

- Solving for $\bar{e}_t$ gives:

$$\bar{e}_t = \frac{\varepsilon \Pi_{1,t}^m}{W_t}. \tag{A.63}$$

  So provided $\Pi_{1,t}^m / W_t = \left(\Pi_{1,t}^m / K_t\right) / (W_t / K_t)$ is stationary (constant steady-state) we find that E_t, $e_t(\eta)$, and $\bar{e}_t$ are stationary also.

- Optimal choices can be written as follows:

$$P_{V,t} c_t^y(\eta) = \frac{1}{1+\beta} HW_t^y(\eta),$$
$$\frac{P_{V,t+1} c_{t+1}^o(\eta)}{1+R_{t+1}^n} = \frac{\beta}{1+\beta} HW_t^y(\eta),$$
$$Q_t \left[z_t^y(\eta) + k_t^y(\eta)\right] = s_t(\eta)\Pi_{1,t}^m + W_t \left[\lambda^y - e_t(\eta)\right] - \frac{1}{1+\beta} HW_t^y(\eta),$$
$$= \frac{\beta}{1+\beta} \left[s_t(\eta)\Pi_{1,t}^m + W_t \left[\lambda^y - e_t(\eta)\right]\right] - \frac{1}{1+\beta} \frac{\lambda^o W_{t+1}}{1+R_{t+1}^n},$$
$$HW_t^y(\eta) \equiv s_t(\eta)\Pi_{1,t}^m + W_t \left[\lambda^y - e_t(\eta)\right] + \frac{\lambda^o W_{t+1}}{1+R_{t+1}^n}.$$

- Aggregate saving (using (A.59), (A.60), and (A.56)):

$$Q_t K_{t+1} = \frac{\beta}{1+\beta}\left[(1-\varepsilon)\Pi^m_{1,t} + \lambda^y W_t\right] - \frac{1}{1+\beta}\frac{\lambda^o W_{t+1}}{1+R^n_{t+1}}. \tag{A.64}$$

- Demand for new capital goods:

$$Q_t Z_t = \frac{\beta}{1+\beta}\left[(1-\varepsilon)\Pi^m_{1,t} + \lambda^y W_t\right] - \frac{1}{1+\beta}\frac{\lambda^o W_{t+1}}{1+R^n_{t+1}} - Q_t(1-\delta)K_t.$$

- Aggregate demands for composite consumption goods:

$$P_{V,t}c^y_t = \frac{1}{1+\beta}HW^y_t,$$
$$\frac{P_{V,t+1}c^o_{t+1}}{1+R^n_{t+1}} = \frac{\beta}{1+\beta}HW^y_t.$$

- Aggregate human wealth of the young (after using noting that $W_t\int e_t(\eta)dF(\eta) = \varepsilon\Pi^m_{1,t}$):

$$HW^y_t \equiv (1-\varepsilon)\Pi^m_{1,t} + \lambda^y W_t + \frac{\lambda^o W_{t+1}}{1+R^n_{t+1}}.$$

- Demand in sector 1 originates from the young and the old.
  - Young demand for good 1:

  $$X^y_{1,t} = \frac{\partial P_{V,t}}{\partial P_{1,t}}c^y_t = \frac{\partial P_{V,t}}{\partial P_{1,t}}\frac{1}{1+\beta}\frac{HW^y_t}{P_{V,t}} = \frac{\alpha^\sigma P^{-\sigma}_{1,t}}{\alpha^\sigma P^{1-\sigma}_{1,t} + (1-\alpha)^\sigma P^{1-\sigma}_{2,t}}\frac{HW^y_t}{1+\beta}.$$

  By holding HW^y_t constant this is interpreted as a Marshallian demand curve.
  - Old demand for good 1:

  $$X^o_{1,t} = \frac{\alpha^\sigma P^{-\sigma}_{1,t}}{\alpha^\sigma P^{1-\sigma}_{1,t} + (1-\alpha)^\sigma P^{1-\sigma}_{2,t}}I^o_t,$$
  $$I^o_t = \lambda^o W_t + \left[(1-\delta)\,Q_t + R^k_t\right]K_t.$$

  - Total demand is thus:

  $$X_{1,t} = \frac{\alpha^\sigma P^{-\sigma}_{1,t}}{\alpha^\sigma P^{1-\sigma}_{1,t} + (1-\alpha)^\sigma P^{1-\sigma}_{2,t}}\left[\frac{HW^y_t}{1+\beta} + I^o_t\right].$$

- The monopolist in sector 1 has the following profit function:

$$\Pi^m_{1,t} = \left[P_{1,t} - MC^x_1(W_t/K_{1,t}, R^k_t)\right]X_{1,t}.$$

and the monopoly price is set according to the usual markup rule:

$$
\begin{aligned}
P_{1,t}^m &= \mu_{1,t}^m MC_1^x(W_t/K_{1,t}, R_t^k), \qquad \mu_{1,t}^m \equiv \frac{\varepsilon_{d,t}^m}{\varepsilon_{d,t}^m - 1} > 1, \\
\varepsilon_{d,t}^m &\equiv -\frac{\partial X_{1,t}}{\partial P_{1,t}}\frac{P_{1,t}}{X_{1,t}} = \frac{\alpha^\sigma (P_{1,t}^m)^{1-\sigma} + \sigma(1-\alpha)^\sigma (P_{2,t}^c)^{1-\sigma}}{\alpha^\sigma (P_{1,t}^m)^{1-\sigma} + (1-\alpha)^\sigma (P_{2,t}^c)^{1-\sigma}}, \\
&= \frac{\alpha^\sigma (p_t^m)^{1-\sigma} + \sigma(1-\alpha)^\sigma}{\alpha^\sigma (p_t^m)^{1-\sigma} + (1-\alpha)^\sigma} > 1, \\
\mu_{1,t}^m &= \frac{\alpha^\sigma (P_{1,t}^m)^{1-\sigma} + \sigma(1-\alpha)^\sigma (P_{2,t}^c)^{1-\sigma}}{(\sigma-1)(1-\alpha)^\sigma (P_{2,t}^c)^{1-\sigma}}, \\
&= \frac{\alpha^\sigma (p_t^m)^{1-\sigma} + \sigma(1-\alpha)^\sigma}{(\sigma-1)(1-\alpha)^\sigma},
\end{aligned}
$$

with:

$$
p_t^m \equiv \frac{P_{1,t}^m}{P_{2,t}^c}.
$$

- For future use we note that:

$$
\mu_{1,t}^m - 1 = \frac{\alpha^\sigma (p_t^m)^{1-\sigma} + (1-\alpha)^\sigma}{(\sigma-1)(1-\alpha)^\sigma}.
$$

- Using the expression for $MC_1^x(W_t/K_{1,t}, R_t^k)$ derived above we find:

$$
P_{1,t}^m = \mu_{1,t}^m P_{2,t}^c mc_1^x(w_t/K_t, r_t^k),
$$

where $mc_1^x(w_t/K_t, r_t^k)$ is real marginal cost in the monopolistic sector:

$$
mc_1^x(w_t/K_t, r_t^k) \equiv \left(\frac{w_t}{\phi K_t}\right)^\phi \left(\frac{r_t^k}{1-\phi}\right)^{1-\phi} \frac{1}{\Omega}.
$$

- It follows that $\varepsilon_{d,t}^m$ can be written as:

$$
\begin{aligned}
\varepsilon_{d,t}^m &= \frac{\alpha^\sigma (\mu_1^m mc_1^x(w_t/K_t, r_t^k) P_{2,t}^c)^{1-\sigma} + \sigma(1-\alpha)^\sigma (P_{2,t}^c)^{1-\sigma}}{\alpha^\sigma (\mu_1^m mc_1^x(w_t/K_t, r_t^k) P_{2,t}^c)^{1-\sigma} + (1-\alpha)^\sigma (P_{2,t}^c)^{1-\sigma}}, \\
&= \frac{\alpha^\sigma \left(\frac{\varepsilon_{d,t}^m}{\varepsilon_{d,t}^m - 1} mc_1^x(w_t/K_t, r_t^k)\right)^{1-\sigma} + \sigma(1-\alpha)^\sigma}{\alpha^\sigma \left(\frac{\varepsilon_{d,t}^m}{\varepsilon_{d,t}^m - 1} mc_1^x(w_t/K_t, r_t^k)\right)^{1-\sigma} + (1-\alpha)^\sigma}.
\end{aligned}
$$

- The elasticity and thus the markup $\mu_{1,t}^m$ depend on $mc_1^x(w_t/K_t, r_t^k) = 1$ and are thus **constant**!

- The derived demands for capital and labour are obtained by employing Shephard's Lemma:

$$H_{1,t}^{m} = \frac{\partial MC_1^x(W_t/K_t, R_t^k)}{\partial W_t} X_{1,t} = \frac{\phi}{W_t}\left(\frac{W_t/K_t}{\phi_1}\right)^{\phi}\left(\frac{R_t^k}{1-\phi}\right)^{1-\phi}\frac{X_{1,t}}{\Omega},$$
$$= \frac{\phi MC_1^x(W_t/K_t, R_t^k)}{W_t} X_{1,t},$$
$$K_{1,t}^{m} = \frac{\partial MC_1^x(W_t/K_t, R_t^k)}{\partial R_t^k} X_{1,t} = \frac{(1-\phi) MC_1^x(W_t/K_t, R_t^k)}{R_t^k} X_{1,t}.$$

- In real terms the factor demand are:

$$r_t^k = (1-\phi) mc_1^x \Omega H_{1,t}^{\phi}\left(\frac{K_{1,t}}{K_t}\right)^{-\phi},$$
$$\frac{w_t}{K_t} = \phi mc_1^x \Omega H_{1,t}^{\phi-1}\left(\frac{K_{1,t}}{K_t}\right)^{1-\phi}.$$

- Aggregate profit equals:

$$\Pi_{1,t}^{m} = X_{1,t}\left(P_{1,t} - MC_1^x(W_t/K_t, R_t^k)\right)$$
$$= \left(\mu_{1,t}^{m} - 1\right) MC_1^x(W_t/K_t, R_t^k) X_{1,t}$$
$$= \Xi_t \left[\frac{HW_t^y}{1+\beta} + I_t^o\right], \tag{A.65}$$

where Ξ_t is an auxiliary term:

$$\Xi_t \equiv \frac{\alpha^{\sigma}\left(p_t^m\right)^{1-\sigma}}{\alpha^{\sigma}\left(p_t^m\right)^{1-\sigma} + \sigma(1-\alpha)^{\sigma}}. \tag{A.66}$$

- We find (after using (A.63)) that:

$$\frac{HW_t^y}{1+\beta} + I_t^o = \frac{1}{1+\beta}\left((1-\varepsilon)\Pi_{1,t}^m + \lambda^y W_t + \frac{\lambda^o W_{t+1}}{1+R_{t+1}^n}\right)$$
$$+ \lambda^o W_t + \left[(1-\delta) Q_t + R_t^k\right] K_t. \tag{A.67}$$

So (as before) current profit depends in part on itself because young agents consume part of it.

- By solving (A.65) and (A.67) for $\Pi_{1,t}^m$ and $\frac{HW_t^y}{1+\beta} + I_t^o$ we find:

$$\Pi_{1,t}^{m} = \frac{\Xi_t}{1+\beta-(1-\varepsilon)\Xi_t}\left[\lambda^y W_t + \frac{\lambda^o W_{t+1}}{1+R_{t+1}^n}\right.$$
$$\left. + (1+\beta)\left[\lambda^o W_t + \left[(1-\delta) Q_t + R_t^k\right] K_t\right]\right]. \tag{A.68}$$

and:

$$\frac{HW_t^y}{1+\beta} + I_t^o = \frac{1}{1+\beta-(1-\varepsilon)\Xi_t}\left[\lambda^y W_t + \frac{\lambda^o W_{t+1}}{1+R_{t+1}^n}\right.$$
$$\left. + (1+\beta)\left[\lambda^o W_t + \left[(1-\delta)\,Q_t + R_t^k\right] K_t\right]\right]. \tag{A.69}$$

- It follows that $\Pi_{1,t}^m$ and $\frac{HW_t^y}{1+\beta} + I_t^o$ are both proportional to the growing variable K_t
- Demand for good 2 originates from the young and the old.
  - Young demand for good 2:

$$X_{2,t}^y = \frac{\partial P_{V,t}}{\partial P_{2,t}} c_t^y = \frac{\partial P_{V,t}}{\partial P_{2,t}} \frac{1}{1+\beta} \frac{HW_t^y}{P_{V,t}} = \frac{(1-\alpha)^\sigma P_{2,t}^{-\sigma}}{\alpha^\sigma P_{1,t}^{1-\sigma} + (1-\alpha)^\sigma P_{2,t}^{1-\sigma}} \frac{HW_t^y}{1+\beta}.$$

  - Old demand for good 1:

$$X_{2,t}^o = \frac{(1-\alpha)^\sigma P_{2,t}^{-\sigma}}{\alpha^\sigma P_{1,t}^{1-\sigma} + (1-\alpha)^\sigma P_{2,t}^{1-\sigma}} I_t^o,$$
$$I_t^o = \lambda^o W_t + \left[(1-\delta)\,Q_t + R_t^k\right] K_t.$$

  - Total demand is thus:

$$X_{2,t} = \frac{(1-\alpha)^\sigma P_{2,t}^{-\sigma}}{\alpha^\sigma P_{1,t}^{1-\sigma} + (1-\alpha)^\sigma P_{2,t}^{1-\sigma}} \left[\frac{HW_t^y}{1+\beta} + I_t^o\right].$$

- Using (A.68) in (A.64) we can write aggregate saving as:

$$Q_t K_{t+1} = \frac{\beta}{1+\beta-(1-\varepsilon)\Xi_t}\left[\lambda^y W_t + (1-\varepsilon)\Xi_t\left[\lambda^o W_t + \left[(1-\delta)\,Q_t + R_t^k\right] K_t\right]\right]$$
$$- \frac{(1+\beta)\left[1-(1-\varepsilon)\Xi_t\right]}{1+\beta-(1-\varepsilon)\Xi_t} \frac{\lambda^o W_{t+1}}{1+R_{t+1}^n}. \tag{A.70}$$

A.5.4.1 Verify Walras Law

- Spending at time t:

$$P_{V,t}\left[c_t^y + c_t^o\right] + Q_t\left[Z_t + (1-\delta)K_t\right] = W_t\left[\lambda^y - \bar{e}_t\right] + \Pi_{1,t}^m + \lambda^o W_t + \left[(1-\delta)Q_t + R_t^k\right] K_t.$$

- The old sell the remaining capital to the young so:

$$P_{V,t}\left[c_t^y + c_t^o\right] + Q_t Z_t = W_t\left[\lambda^y - \bar{e}_t\right] + \Pi_{1,t}^m + \lambda^o W_t + R_t^k K_t.$$

- But $P_{V,t}\left[c_t^y + c_t^o\right] = P_{1,t}X_{1,t} + P_{2,t}X_{2,t}$ and $H_t = \lambda^y + \lambda^o - \bar{e}_t$ so we get:

$$P_{1,t}X_{1,t} + P_{2,t}X_{2,t} + Q_t Z_t = W_t H_t + \Pi_{1,t}^m + R_t^k K_t.$$

- But $\Pi_{1,t}^m = (P_{1,t} - MC_1^x(W_t/K_{1,t}, R_t^k))X_{1,t}$, $P_{2,t} = MC_2^x(W_t/K_{2,t}, R_t^k)$, and $Q_t = MC^z(W_t/K_{z,t}, R_t^k)$ so we get:

$$MC_1^x(W_t/K_{1,t}, R_t^k)X_{1,t} + MC_2^x(W_t/K_{2,t}, R_t^k)X_{2,t} + MC^z(W_t/K_{z,t}, R_t^k)Z_t = W_t H_t + R_t^k K_t.$$

Right-hand side: total factor income. Left-hand side: total spending on consumption and investment goods evaluated at the true marginal cost of producing these goods.

A.5.4.2 Checking market equilibrium conditions

- Market for good 1 (demand and supply):

$$\begin{aligned}\frac{X_{1,t}}{K_t} &= \frac{\alpha^\sigma p_{1,t}^{-\sigma}}{\alpha^\sigma p_{1,t}^{1-\sigma} + (1-\alpha)^\sigma}\left[\frac{1}{1+\beta}\left(\frac{(1-\varepsilon)\pi_{1,t}^m + \lambda^y w_t}{K_t} + \lambda^o \frac{1+\gamma_{t+1}}{1+r_{t+1}}\frac{w_{t+1}}{K_{t+1}}\right)\right.\\ &\quad \left. + \lambda^o \frac{w_t}{K_t} + \left[(1-\delta)\, q_t + r_t^k\right]\right],\\ \frac{X_{1,t}}{K_t} &= \Omega H_{1,t}^\phi \left(\frac{K_{1,t}}{K_t}\right)^{1-\phi}.\end{aligned}$$

- Market for good 2 (demand and supply):

$$\begin{aligned}\frac{X_{2,t}}{K_t} &= \frac{(1-\alpha)^\sigma}{\alpha^\sigma p_{1,t}^{1-\sigma} + (1-\alpha)^\sigma}\left[\frac{1}{1+\beta}\left(\frac{(1-\varepsilon)\pi_{1,t}^m + \lambda^y w_t}{K_t} + \lambda^o \frac{1+\gamma_{t+1}}{1+r_{t+1}}\frac{w_{t+1}}{K_{t+1}}\right)\right.\\ &\quad \left. + \lambda^o \frac{w_t}{K_t} + \left[(1-\delta)\, q_t + r_t^k\right]\right],\\ \frac{X_{2,t}}{K_t} &= \Omega H_{2,t}^\phi \left(\frac{K_{2,t}}{K_t}\right)^{1-\phi}.\end{aligned}$$

- Market for investment goods (demand and supply):

$$\begin{aligned}q_t \frac{Z_t}{K_t} &= \frac{\beta}{1+\beta}\left[\frac{(1-\varepsilon)\pi_{1,t}^m + \lambda^y w_t}{K_t}\right] - \frac{1}{1+\beta}\lambda^o \frac{1+\gamma_{t+1}}{1+r_{t+1}}\frac{w_{t+1}}{K_{t+1}} - q_t(1-\delta)\\ \frac{Z_t}{K_t} &= \Omega H_{z,t}^\phi \left(\frac{K_{z,t}}{K_t}\right)^{1-\phi}.\end{aligned}$$

- Aggregate output:

$$\frac{Y_t}{K_t} = p_t \frac{X_{1,t}}{K_t} + \frac{X_{2,t}}{K_t} + q_t \frac{Z_t}{K_t}.$$

A.5.5 Recalibration

- Model must be recalibrated to yield an 'observationally equivalent' competitive steady-state growth path. Otherwise we are comparing apples with oranges. Calibrate sequentially, starting with a one-sector version of the model

- One-sector model: $\phi_1 = \phi_2 = \psi = \phi$, $\Omega_1 = \Omega_2 = \Omega_z = \Omega$, $\bar{h} = 1$, and $\alpha = 0.5$.

  - Dynamic model:

$$
\begin{aligned}
\frac{K_{t+1}}{K_t} &= \frac{\bar{h}}{1+\beta}\left[\beta\lambda^y \frac{w_t}{K_t} - \frac{\lambda^o}{1+r_{t+1}}\frac{w_{t+1}}{K_{t+1}}\frac{K_{t+1}}{K_t}\right], \\
r_t &= (1-\phi)\,\Omega H_t^{\phi} - \delta, \\
\frac{w_t}{K_t} &= \phi\Omega H_t^{\phi-1}, \\
\frac{Y_t}{K_t} &= \Omega H_t^{\phi}, \\
H_t &= \lambda^y + \lambda^o, \\
\frac{I_t}{K_t} &= \frac{K_{t+1}}{K_t} - (1-\delta).
\end{aligned}
$$

  - Features of the steady-state growth path:

$$
\begin{aligned}
1+\gamma^* &= \frac{\bar{h}}{1+\beta}\left[\beta\lambda^y - \lambda^o\frac{1+\gamma^*}{1+r^*}\right]\left(\frac{w_t}{K_t}\right)^*, \\
r^* &= (1-\phi)\,\Omega(\lambda^y+\lambda^o)^{\phi} - \delta, \\
\left(\frac{w_t}{K_t}\right)^* &= \phi\Omega(\lambda^y+\lambda^o)^{\phi-1} \\
\left(\frac{Y_t}{K_t}\right)^* &= \Omega(\lambda^y+\lambda^o)^{\phi}, \\
\left(\frac{I_t}{K_t}\right)^* &= \gamma^* + \delta.
\end{aligned}
$$

- Numerical content one-sector model:

  - We set parameters:

$$
\alpha = 0.5, \qquad \lambda^y = 0.9, \qquad \lambda^o = 0.5, \qquad \delta = 0.8437, \qquad \phi_1 = \phi_2 = \psi = 0.75, \qquad T = 30.
$$

  - We set targets:

$$
\gamma_a^* = 0.025, \qquad r_a^* = 0.05.
$$

– We know that:

$$\begin{aligned}
r^* &= (1+r_a^*)^T - 1 = 3.3219,\\
\gamma^* &= (1+\gamma_a^*)^T - 1 = 1.0976\\
(r^k)^* &= r^* + \delta = 4.1657,\\
y^* &= \frac{r^*+\delta}{1-\phi} = 16.6627,\\
z^* &= \gamma^* + \delta = 1.9413,\\
x^* &= y^* - z^* = 14.7214,\\
\left(\frac{w_t}{K_t}\right)^* &= \frac{\phi y^*}{\lambda^y + \lambda^o} = 8.9264.
\end{aligned}$$

– Note that $(X/Y)^* = 0.8835$ and $(Z/Y)^* = 0.1165$.

– We choose β and Ω to make it fit the model:

$$\beta = 0.6608, \qquad \Omega = 12.2936.$$

- Special case of the three-sector model: $\phi_1 = \phi_2 = \psi = \phi$ and $\Omega_1 = \Omega_2 = \Omega_z = \Omega$.

  – General model: see Table A.7

  – Steady-state competitive growth path features $p^* = q^* = 1$ and $\bar{e}^* = (\pi_1^m)^* = 0$ so that:

$$\begin{aligned}
1+\gamma^* &= \frac{1}{1+\beta}\left(\frac{w}{K}\right)^*\left[\beta\lambda^y - \lambda^o\frac{1+\gamma^*}{1+r^*}\right]\\
(r^k)^* &= r^* + \delta\\
\left(\frac{w}{K}\right)^* &= \phi\Omega\left(\frac{H_1^*}{u_1^*}\right)^{\phi-1} = \phi\Omega\left(\frac{H_2^*}{u_2^*}\right)^{\phi-1} = \phi\Omega\left(\frac{H_z^*}{u_z^*}\right)^{\phi-1}\\
(r^k)^* &= (1-\phi)\Omega\left(\frac{H_1^*}{u_1^*}\right)^{\phi} = (1-\phi)\Omega\left(\frac{H_2^*}{u_2^*}\right)^{\phi} = (1-\phi)\Omega\left(\frac{H_z^*}{u_z^*}\right)^{\phi}\\
H^* &= H_1^* + H_2^* + H_z^*\\
z^* &= \gamma^* + \delta\\
H^* &= \lambda^y + \lambda^o\\
y^* &= x_1^* + x_2^* + z^*\\
x_1^* &= \frac{\alpha^\sigma}{\alpha^\sigma + (1-\alpha)^\sigma}\left[\frac{1}{1+\beta}\left(\frac{w}{K}\right)^*\left(\lambda^y + \lambda^o\frac{1+\gamma^*}{1+r^*}\right)\right.\\
&\quad\left. + \lambda^o\left(\frac{w}{K}\right)^* + 1 - \delta + (r^k)^*\right]\\
x_1^* &= \Omega(H_1^*)^\phi(u_1^*)^{1-\phi}
\end{aligned}$$

$$
\begin{aligned}
x_2^* &= \Omega(H_2^*)^{\phi}(u_2^*)^{1-\phi} \\
z^* &= \Omega(H_z^*)^{\phi}(u_z^*)^{1-\phi} \\
1 &= u_1^* + u_2^* + u_z^*
\end{aligned}
$$

- Calibrations steps:

  – Keep

$$
\begin{aligned}
r^* &= 3.3219, \\
\gamma^* &= 1.0976 \\
(r^k)^* &= r^* + \delta = 4.1657, \\
y^* &= 16.6627, \\
x_1^* = x_2^* &= \frac{14.7214}{2}, \\
z^* &= \gamma^* + \delta = 1.9413, \\
\left(\frac{w}{K}\right)^* &= 8.9265.
\end{aligned}
$$

  – Define the wage-rental ratio, $\xi^* \equiv (w_t/(r_t^k K_t))^*$ and note:

$$
\frac{H_i^*}{u_i^*} = \frac{\phi}{1-\phi}\frac{1}{\xi^*}, \qquad (1+\lambda^o)\xi^* = \frac{\phi}{1-\phi} \qquad \Leftrightarrow \qquad \phi = 0.75.
$$

  – Hence:

$$
\frac{H_i^*}{u_i^*} = \lambda^y + \lambda^o = 1.4.
$$

  – It follows that:

$$
\begin{aligned}
x_1^* &= \Omega\left(\frac{H_1^*}{u_1^*}\right)^{\phi} u_1^* = \Omega\left(\lambda^y + \lambda^o\right)^{\phi} u_1^*, \\
x_2^* &= \Omega\left(\frac{H_2^*}{u_2^*}\right)^{\phi} u_2^* = \Omega\left(\lambda^y + \lambda^o\right)^{\phi} u_2^*, \\
z^* &= \Omega\left(\frac{H_z^*}{u_z^*}\right)^{\phi} u_z^* = \Omega\left(\lambda^y + \lambda^o\right)^{\phi} u_z^*,
\end{aligned}
$$

  so that:

$$
y^* = \Omega\left(\lambda^y + \lambda^o\right)^{\phi}.
$$

  – Hence Ω is:

$$
\Omega = y^*\left(\lambda^y + \lambda^o\right)^{-\phi} = 12.9464.
$$

– To determine u_z^*, ans $u_1^* = u_2^* = (1 - u_z^*)/2$ we note that:

$$[z^* =] \ \gamma^* + \delta = \Omega \left(\lambda^y + \lambda^o\right)^\phi u_z^* \qquad \Leftrightarrow \qquad u_z^* = \frac{\gamma^* + \delta}{y^*} = 0.1165.$$

– Hence:

$$u_1^* = u_2^* = \frac{1 - u_z^*}{2} = 0.4417.$$

– Finally, we choose β to ensure that the followsing relationship is satisfied:

$$1 + \gamma^* = \frac{1}{1+\beta} \left[\beta \lambda^y - \lambda^o \frac{1+\gamma^*}{1+r^*} \right] \left(\frac{w}{K} \right)^* \qquad \Leftrightarrow$$
$$\beta = \frac{1+\gamma^*}{\lambda^y (w/K)^* - (1+\gamma^*)} \left[1 + \frac{\lambda^o (w/K)^*}{1+r^*} \right] = 0.7182.$$

- To summarize, the structural parameters are as given in Table 3 with the following exceptions:

$$\beta = 0.7182, \quad \phi_1 = \phi_2 = \psi = 0.75, \quad \Omega_1 = \Omega_2 = \Omega_z = 12.9464, \quad \phi_e = \theta = 0.$$

- See Table A.7 for a full listing and Table 6 in the paper for a compact listing of the capital-externality model with rent-seeking.
- See Table A.8 for the quantitative steady-state results.

Table A.7: Rent-seeking and growth with a physical capital externality

$$q_t(1+\gamma_{t+1}) = \frac{1}{1+\beta}\left[\beta\frac{(1-\varepsilon)\pi^m_{1,t}+\lambda^y w_t}{K_t} - \lambda^o \frac{1+\gamma_{t+1}}{1+r_{t+1}}\frac{w_{t+1}}{K_{t+1}}\right] \tag{AT7.1}$$

$$\frac{\pi^m_{1,t}}{K_t} = \frac{\Xi}{1+\beta-(1-\varepsilon)\Xi}\left[\lambda^y\frac{w_t}{K_t} + \lambda^o\frac{1+\gamma_{t+1}}{1+r_{t+1}}\frac{w_{t+1}}{K_{t+1}}\right] + \frac{(1+\beta)\Xi}{1+\beta-(1-\varepsilon)\Xi}\left[\lambda^o\frac{w_t}{K_t}+(1-\delta)q_t + r^k_t\right] \tag{AT7.2}$$

$$\frac{w_t}{K_t}\bar{e}_t = \varepsilon\frac{\pi^m_{1,t}}{K_t} \tag{AT7.3}$$

$$1+r_{t+1} \equiv \frac{r^k_{t+1}+(1-\delta)q_{t+1}}{q_t} \tag{AT7.4}$$

$$\frac{w_t}{K_t} = \phi mc^x_{1,t}\Omega\left(\frac{H_{1,t}}{u_{1,t}}\right)^{\phi-1} = \phi\Omega\left(\frac{H_{2,t}}{u_{2,t}}\right)^{\phi-1} = \phi q_t\Omega\left(\frac{H_{z,t}}{u_{z,t}}\right)^{\phi-1} \qquad \text{(AT7.5)-(AT7.7)}$$

$$r^k_t = (1-\phi)mc^x_{1,t}\Omega\left(\frac{H_{1,t}}{u_{1,t}}\right)^{\phi} = (1-\phi)\Omega\left(\frac{H_{2,t}}{u_{2,t}}\right)^{\phi} = (1-\phi)q_t\Omega\left(\frac{H_{z,t}}{u_{z,t}}\right)^{\phi} \qquad \text{(AT7.8)-(AT7.10)}$$

$$H_t = H_{1,t}+H_{2,t}+H_{z,t} \tag{AT7.11}$$

$$z_t = \gamma_{t+1}+\delta \tag{AT7.12}$$

$$H_t = \lambda^o+\lambda^y-\bar{e}_t \tag{AT7.13}$$

$$y_t = px_{1,t}+x_{2,t}+q_t z_t \tag{AT7.14}$$

$$\Xi \equiv \frac{\alpha^\sigma p^{1-\sigma}}{\alpha^\sigma p^{1-\sigma}+\sigma(1-\alpha)^\sigma} \tag{AT7.15}$$

$$p = \frac{\alpha^\sigma p^{1-\sigma}+\sigma(1-\alpha)^\sigma}{(\sigma-1)(1-\alpha)^\sigma}mc^x_{1,t} \tag{AT7.16}$$

$$px_{1,t} = \frac{\alpha^\sigma p^{1-\sigma}}{\alpha^\sigma p^{1-\sigma}+(1-\alpha)^\sigma}\left[\frac{1}{1+\beta}\left(\frac{(1-\varepsilon)\pi^m_{1,t}+\lambda^y w_t}{K_t}+\lambda^o\frac{1+\gamma_{t+1}}{1+r_{t+1}}\frac{w_{t+1}}{K_{t+1}}\right) + \lambda^o\frac{w_t}{K_t}+(1-\delta)q_t+r^k_t\right] \tag{AT7.17}$$

$$x_{1,t} = \Omega H^\phi_{1,t}u^{1-\phi}_{1,t} \tag{AT7.18}$$

$$x_{2,t} = \Omega H^\phi_{2,t}u^{1-\phi}_{2,t} \tag{AT7.19}$$

$$z_t = \Omega H^\phi_{z,t}u^{1-\phi}_{z,t} \tag{AT7.20}$$

$$1 = u_{1,t}+u_{2,t}+u_{z,t} \tag{AT7.21}$$

Notes The endogenous variables are $\gamma_{t+1} \equiv (K_{t+1}-K_t)/K_t$, $\bar{e}_t$, $\pi^m_{1,t}/K_t$, r_t, q_t, r^k_t, w_t/K_t, $x_{1,t} \equiv X_{1,t}/K_t$, $x_{2,t} \equiv X_{2,t}/K_t$, $z_t \equiv Z_t/K_t$, $u_{1,t} \equiv K_{1,t}/K_t$, $u_{2,t} \equiv K_{2,t}/K_t$, $u_{z,t} \equiv K_{z,t}/K_t$, $H_{1,t}$, $H_{2,t}$, $H_{z,t}$, H_t, $mc^x_{1,t}$, Ξ, p, and $y_t \equiv Y_t/K_t$.

Table A.8: Features of the steady-state growth path (KE case)

	(a)	(b)	(c)	(d)	(e)	(f)
ε		0.0800	<u>0.1600</u>	0.0800	0.0800	0.0800
σ	2.0000	2.0000	2.0000	<u>4.0000</u>	2.0000	2.0000
α	0.5000	0.5000	0.5000	0.5000	<u>0.7000</u>	0.5000
ϕ	0.7500	0.7500	0.7500	0.7500	0.7500	<u>0.6000</u>
y^*	16.6627	19.2956	19.0425	17.7584	25.0668	18.3257
x_1^*	7.3607	2.0232	2.0015	2.6818	3.5961	1.9077
x_2^*	7.3607	11.7918	11.6657	11.6611	8.2706	11.1190
i^*	1.9413	2.6194	2.5447	2.2247	4.0713	2.6010
e^*		0.0255	0.0503	0.0107	0.0806	0.0315
γ^*	1.0976	1.7757	1.7009	1.3810	3.2275	1.7573
$\gamma_a^* \times 100\%$	2.5000	3.4616	3.3674	2.9339	4.9227	3.4386
$\gamma_{ca}^* \times 100\%$	2.5000					2.2658
$(w/K)^*$	8.9265	8.9676	9.0085	8.9435	9.0598	6.8518
$(r^k)^*$	4.1657	4.1086	4.0530	4.1419	3.9845	6.2511
r^*	3.3219	3.2649	3.2092	3.2981	3.1407	5.4074
$r_a^* \times 100\%$	5.0000	4.9535	4.9076	4.9807	4.8502	6.3872
p^*	1.0000	2.4142	2.4142	1.4440	3.5386	2.4142
$(mc^x)^*$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
q^*	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
u_1^*	0.4417	0.1231	0.1235	0.1619	0.2256	0.1221
u_2^*	0.4417	0.7175	0.7196	0.7038	0.5189	0.7115
u_z^*	0.1165	0.1594	0.1570	0.1343	0.2554	0.1664
H^*	1.4000	1.3745	1.3497	1.3893	1.3194	1.3685
H_1^*	0.6184	0.1692	0.1666	0.2249	0.2977	0.1671
H_2^*	0.6184	0.9862	0.9712	0.9779	0.6847	0.9737
H_z^*	0.1631	0.2191	0.2119	0.1866	0.3370	0.2278
$(\pi_1^m/K)^*$	0.0000	2.8612	2.8306	1.1908	9.1289	2.6979

Notes The perfectly competitive steady-state equilibrium (without rent seeking) is reported in column (a). Column (b) reports on the benchmark rent-seeking equilibrium. Columns (c)–(f) report on some alternative rent-seeking equilibria for different values of, respectively, ε, σ, α, and ϕ,

A.6 Welfare analysis

- In order to understand the welfare effects of rent seeking we must characterize the first-best social optimum.
- The model is slightly complicated because:
  - There are overlapping generations that need to be weighted in the appropriate manner.
  - There are external effects due to human capital accumulation.
- Social welfare function is conform the insights of Calvo and Obstfeld (1988):

$$SW_t = \frac{1}{\omega}\Lambda_{t-1}^y + \Lambda_t^y + \omega\Lambda_{t+1}^y + \omega^2\Lambda_{t+2}^y + \ldots = \sum_{\tau=t-1}^{\infty} \Lambda_\tau^y \omega^{\tau-t}$$

with:

$$\Lambda_\tau^y \equiv \int_{\eta_L}^{\eta_H} \Lambda_\tau^y(\eta) dF(\eta)$$

and:

$$\Lambda_t^y(\eta) \equiv \ln c_t^y(\eta) + \beta \ln c_{t+1}^o(\eta)$$
$$c_t^s(\eta) \equiv \left[\alpha x_{1,t}^s(\eta)^{1-1/\sigma} + (1-\alpha) x_{2,t}^s(\eta)^{1-1/\sigma}\right]^{1/(1-1/\sigma)}, \qquad \text{(for } s \in \{y,o\}\text{)}$$

- The social planner treats agents of differing rent-seeking ability symmetrically. This means that:
  - We can impose symmetry up front:

$$c_t^s(\eta) = c_t^s, \quad l_t^s(\eta) = l_t^s, \quad x_{i,t}^s(\eta) = x_{1,t}^s, \qquad \text{(for all } t \text{ and } \eta,\ s \in \{y,o\}\text{)}$$

  - There is no rent seeking:

$$e_t(\eta) = 0$$

- The constraints consist of:
  - Technology:

$$X_{i,t} = \Phi^x(H_{i,t}, K_{i,t})$$
$$Z_t = \Phi^z(H_{z,t}, K_{z,t})$$

- Total demands for the goods:

$$X_{i,t} = \int_{\eta_L}^{\eta_H} \left[x_{i,t}^y(\eta) + x_{i,t}^o(\eta)\right] dF(\eta)$$

- Resources:

$$K_t = K_{1,t} + K_{2,t} + K_{z,t}$$
$$H_t = H_{1,t} + H_{2,t} + H_{z,t}$$
$$H_t = \int_{\eta_L}^{\eta_H} \left[\lambda h_t^o(\eta) + [1 - l_t(\eta)]\, h_t^y(\eta)\right] dF(\eta)$$

- Accumulation:

$$h_{t+1}^o(\eta) = h_t^y(\eta)\left[1 + \phi_e \frac{l_t(\eta)^{1-\theta}}{1-\theta}\right]$$
$$K_{t+1} = Z_t + (1-\delta)K_t$$

- Initial condition for the young:

$$h_t^y(\eta) = \bar{h}_t \equiv \int_{\eta_L}^{\eta_H} h_t^o(\eta) dF(\eta)$$

- Lagrangian:

$$\begin{aligned}
\mathcal{L}_t \equiv\ & \frac{1}{\omega}\left[\ln c_{t-1}^y + \beta \ln c_t^o\right] + \ln c_t^y + \beta \ln c_{t+1}^o + \dots \\
& + \sum_{\tau=t}^{\infty} \Big\{ \lambda_{1,\tau}\left[\Phi^x(H_{1,\tau}, K_{1,\tau}) - x_{1,\tau}^y - x_{1,\tau}^o\right] + \lambda_{2,\tau}\left[\Phi^x(H_{2,\tau}, K_{2,\tau}) - x_{2,\tau}^y - x_{2,\tau}^o\right] \\
& + \lambda_{3,\tau}\left[K_\tau - K_{1,\tau} - K_{2,\tau} - K_{z,\tau}\right] + \lambda_{4,\tau}\left[(1 + \lambda - l_t)\bar{h}_\tau - H_{1,\tau} - H_{2,\tau} - H_{z,\tau}\right] \\
& + \lambda_{5,\tau}\left[\bar{h}_\tau\left(1 + \phi_e \frac{l_\tau^{1-\theta}}{1-\theta}\right) - \bar{h}_{\tau+1}\right] + \lambda_{6,\tau}\left[Z_\tau + (1-\delta)K_\tau - K_{\tau+1}\right] \\
& + \lambda_{7,\tau}\left[\Phi^z(H_{z,\tau}, K_{z,\tau}) - Z_\tau\right] \Big\}
\end{aligned}$$

- Taken as given at time t are c_{t-1}^y, K_t, and $\bar{h}_t$
- FONCs for period t:
  - Consumption components:

$$\frac{\partial \mathcal{L}_t}{\partial x_{i,t}^y} = \frac{1}{c_t^y}\frac{\partial c_t^y}{\partial x_{i,t}^y} - \lambda_{i,t} = 0 \qquad \text{(for } i = 1, 2\text{)}$$
$$\frac{\partial \mathcal{L}_t}{\partial x_{i,t}^o} = \frac{\beta}{\omega c_t^o}\frac{\partial c_t^o}{\partial x_{i,t}^o} - \lambda_{i,t} = 0 \qquad \text{(for } i = 1, 2\text{)}$$

– Factor usage:

$$\frac{\partial \mathcal{L}_t}{\partial H_{1,t}} = \lambda_{1,t} \frac{\partial \Phi^x(H_{1,\tau}, K_{1,\tau})}{\partial H_{1,t}} - \lambda_{4,t} = 0$$
$$\frac{\partial \mathcal{L}_t}{\partial H_{2,t}} = \lambda_{2,t} \frac{\partial \Phi^x(H_{2,\tau}, K_{2,\tau})}{\partial H_{2,t}} - \lambda_{4,t} = 0$$
$$\frac{\partial \mathcal{L}_t}{\partial H_{z,t}} = \lambda_{7,t} \frac{\partial \Phi^z(H_{z,\tau}, K_{z,\tau})}{\partial H_{z,t}} - \lambda_{4,t} = 0$$
$$\frac{\partial \mathcal{L}_t}{\partial K_{1,t}} = \lambda_{1,t} \frac{\partial \Phi^x(H_{1,\tau}, K_{1,\tau})}{\partial K_{1,t}} - \lambda_{3,t} = 0$$
$$\frac{\partial \mathcal{L}_t}{\partial K_{2,t}} = \lambda_{2,t} \frac{\partial \Phi^x(H_{2,\tau}, K_{2,\tau})}{\partial K_{2,t}} - \lambda_{3,t} = 0$$
$$\frac{\partial \mathcal{L}_t}{\partial K_{z,t}} = \lambda_{7,t} \frac{\partial \Phi^z(H_{z,\tau}, K_{z,\tau})}{\partial K_{z,t}} - \lambda_{3,t} = 0$$

- Investment in physical and human capital (schooling):

$$\frac{\partial \mathcal{L}_t}{\partial Z_t} = \lambda_{6,t} - \lambda_{7,t} = 0$$
$$\frac{\partial \mathcal{L}_t}{\partial l_t} = \left[\lambda_{5,t} \phi_e l_t^{-\theta} - \lambda_{4,t}\right] \bar{h}_t = 0$$

- Accumulation:

$$\frac{\partial \mathcal{L}_t}{\partial K_{t+1}} = -\lambda_{6,t} + \lambda_{6,t+1}(1-\delta) + \lambda_{3,t+1} = 0$$
$$\frac{\partial \mathcal{L}_t}{\partial \bar{h}_{t+1}} = -\lambda_{5,t} + \lambda_{4,t+1}(1+\lambda - l_{t+1}) + \lambda_{5,t+1}\left(1 + \phi_e \frac{l_{t+1}^{1-\theta}}{1-\theta}\right) = 0$$

- Summary of the first-order conditions:

$$\frac{\lambda_{1,t}}{\lambda_{2,t}} = \frac{\alpha}{1-\alpha}\left(\frac{x_{2,t}^s}{x_{1,t}^s}\right)^{1/\sigma}, \qquad (\text{for } s \in \{y, o\})$$
$$\frac{\lambda_{1,t}}{\lambda_{2,t}} = \frac{\partial \Phi^x(H_{2,\tau}, K_{2,\tau})/\partial H_{2,t}}{\partial \Phi^x(H_{1,\tau}, K_{1,\tau})/\partial H_{1,t}} = \frac{\partial \Phi^x(H_{2,\tau}, K_{2,\tau})/\partial K_{2,t}}{\partial \Phi^x(H_{1,\tau}, K_{1,\tau})/\partial K_{1,t}}$$
$$\frac{\lambda_{1,t}}{\lambda_{7,t}} = \frac{\partial \Phi^z(H_{z,\tau}, K_{z,\tau})/\partial H_{2,t}}{\partial \Phi^x(H_{1,\tau}, K_{1,\tau})/\partial H_{1,t}} = \frac{\partial \Phi^z(H_{z,\tau}, K_{z,\tau})/\partial K_{z,t}}{\partial \Phi^x(H_{1,\tau}, K_{1,\tau})/\partial K_{1,t}}$$
$$\lambda_{6,t} = \lambda_{6,t+1}(1-\delta) + \lambda_{3,t+1} = 0$$
$$\lambda_{5,t} = \lambda_{5,t+1}\left[\phi_e l_{t+1}^{-\theta}(1+\lambda - l_{t+1}) + \left(1 + \phi_e \frac{l_{t+1}^{1-\theta}}{1-\theta}\right)\right]$$

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