Foundations of Modern Macroeconomics
Third Edition

Chapter 17: Decision making in a stochastic environment

Ben J. Heijdra

Department of Economics, Econometrics & Finance
University of Groningen

13 December 2016

Outline

Aims of this chapter (1)

- Introduce the method of Dynamic Programming
  - Focus on a number of simple consumption-savings examples
  - Once these examples are well understood, take the nontrivial step from a deterministic to a stochastic world
- Introduce the concept of complete markets
  - Arrow-Debreu securities
  - Negishi's insight
  - Competitive risk sharing
- Construct the representative agent under complete markets
  - A-D securities and aggregation
  - Macroeconomic irrelevance of heterogeneity

Three-period consumption example (1)

- Individual who lives for three periods and has the following lifetime utility function:

$$\Lambda_1 \equiv U(C_1) + \beta U(C_2) + \beta^2 U(C_3) \tag{S1}$$

  - C_t is consumption in period t
  - $\beta \equiv 1/(1+\rho)$ is the discount factor due to impatience (ρ is the rate of time preference, $\rho > 0$)
  - $U(x)$ is a felicity function satisfying $U'(x) > 0$, $U''(x) < 0$, and the usual Inada style condition $\lim_{x\to 0} U'(x) = +\infty$

- Felicity function is logarithmic:

$$U(C_t) = \ln C_t \tag{S2}$$

Three-period consumption example (2)

- Financial asset accumulation:

$$A_{t+1} = (1 + r_t)A_t + w_t - C_t \qquad \text{(S3)}$$

  - r_t is the interest rate in period t
  - w_t is the wage in period t
  - A_t is assets at the start of period t.
  - Initial stock of financial assets A_1 at time $t = 1$ (savings from the past)

- The agent chooses C_t and A_{t+1} for $t \in \{1, 2, 3\}$ to maximize (S1) subject to (S3), taking as given (a) initial assets A_1 and (b) the paths of factor prices r_t and w_t
- Since the world ends for this consumer at the end of period $t = 3$ there is a terminal constraint of the form:

$$A_{t+4} \geq 0 \qquad \text{(S4)}$$

Traditional solution method (1)

- Lagrangian:

$$\begin{aligned}\mathcal{L}_1 \equiv\ & U(C_1) + \beta U(C_2) + \beta^2 U(C_3) \\ & + \lambda_1 \left[(1+r_1)A_1 + w_1 - C_1 - A_2\right] \\ & + \lambda_2 \left[(1+r_2)A_2 + w_2 - C_2 - A_3\right] \\ & + \lambda_3 \left[(1+r_3)A_3 + w_3 - C_2 - A_4\right]\end{aligned}$$

where λ_t are the Lagrange multipliers

- First-order necessary conditions for consumption:

$$\frac{\partial \mathcal{L}_1}{\partial C_1} = U'(C_1) - \lambda_1 = 0 \qquad \text{(S5a)}$$

$$\frac{\partial \mathcal{L}_1}{\partial C_2} = \beta U'(C_2) - \lambda_2 = 0 \qquad \text{(S5b)}$$

$$\frac{\partial \mathcal{L}_1}{\partial C_3} = \beta^2 U'(C_3) - \lambda_3 = 0 \qquad \text{(S5c)}$$

Traditional solution method (2)

- First-order necessary conditions for consumption:

$$\frac{\partial \mathcal{L}_1}{\partial A_2} = -\lambda_1 + (1+r_2)\lambda_2 = 0 \quad \text{(S5d)}$$

$$\frac{\partial \mathcal{L}_1}{\partial A_3} = -\lambda_2 + (1+r_3)\lambda_3 = 0 \quad \text{(S5e)}$$

$$\frac{\partial \mathcal{L}_1}{\partial A_4} = -\lambda_3 \leq 0, \qquad A_4 \geq 0, \qquad A_4 \frac{\partial \mathcal{L}_1}{\partial A_4} = 0 \quad \text{(S5f)}$$

- Since $\lambda_3 > 0$, it follows from (S5f) that the consumer will exhaust all his financial assets during the last period of life:

$$A_4^* = 0$$

where the star designates the *optimum choice* for A_4

Traditional solution method (3)

- Consolidated lifetime budget constraint:

$$(1+r_1)A_1 + H_1 = C_1 + \frac{C_2}{1+r_2} + \frac{C_3}{(1+r_2)(1+r_3)} \tag{S6}$$

where H_1 is human wealth:

$$H_1 \equiv w_1 + \frac{w_2}{1+r_2} + \frac{w_3}{(1+r_2)(1+r_3)} \tag{S7}$$

- Using (S5a)–(S5c) and (S2) in (S6) we find:

$$C_1^* = \frac{(1+r_1)A_1 + H_1}{1+\beta+\beta^2} \tag{S8a}$$

$$\frac{C_2^*}{1+r_2} = \beta\frac{(1+r_1)A_1 + H_1}{1+\beta+\beta^2} \tag{S8b}$$

$$\frac{C_3^*}{(1+r_2)(1+r_3)} = \beta^2\frac{(1+r_1)A_1 + H_1}{1+\beta+\beta^2} \tag{S8c}$$

Traditional solution method (4)

- Optimal asset levels:

$$A_2^* = (1 + r_1)A_1 + w_1 - C_1^* \tag{S8d}$$
$$A_3^* = (1 + r_2)A_2 + w_2 - C_2^* \tag{S8e}$$
$$A_4^* = 0 \tag{S8f}$$

- Parameterized version of the three-period model in **Table 17.1**
  - Each period is 25 years; zero initial financial assets, i.e. $A_1 = 0$
  - The wage rate and interest rate are both constant over time, i.e. $r_t = r$ and $w_t = w$
  - Output per worker is normalized to unity so that with a capital share of $\alpha = 0.3$ the wage rate is equal to $w = 0.7$
  - For annual $r^a = 0.04$ and $\rho^a = 0.03$ we find $r = 1.6658$ and $\beta = 0.4776$
  - Panel (a) of **Table 17.1** shows that the consumer is a strong saver in the first two periods and a dissaver in the final period

Table 17.1: Some numerical examples

(a) **Deterministic choices**

Consumption:	C_1^*	0.6221
	C_2^*	0.7920
	C_3^*	1.0084
Assets:	A_2^*	0.0779
	A_3^*	0.1157
	A_4^*	0.0000

Alternative viewpoint (1)

- There is an alternative way of writing down the solutions which gives us a first glance at *policy functions*
- Consider the consumer who has a in assets in period $t = 1$. What does he choose for current consumption and next period's assets?
- No need to redo the optimization problem
- Substitute a for A_1 in (S8a) and (S8b):

$$\begin{aligned}\hat{C}_1 &= \mathbf{C}_1(a; r_1, r_2, r_2, w_1, w_2, w_3) \\ &\equiv \frac{1}{1+\beta+\beta^2}\left[(1+r_1)a + w_1 + \frac{w_2}{1+r_2} + \frac{w_3}{(1+r_2)(1+r_3)}\right]\end{aligned} \tag{S9a}$$

Alternative viewpoint (2)

- and:

$$
\begin{aligned}
\hat{A}_2 &= (1+r_1)a + w_1 - \hat{C}_1 \\
&= \mathbf{A}_1^{+}(a; r_1, r_2, r_3, w_1, w_2, w_3) \\
&\equiv \frac{\beta(1+\beta)}{1+\beta+\beta^2}\left[(1+r_1)a + w_1\right] \\
&\quad - \frac{1}{1+\beta+\beta^2}\left[\frac{w_2}{1+r_2} + \frac{w_3}{(1+r_2)(1+r_3)}\right] \qquad \text{(S9b)}
\end{aligned}
$$

where the hats designate *conditionally optimal* choices

Alternative viewpoint (3)

- The *policy function* $\mathbf{C}_1(a;\cdot)$ in (S9a) gives the choice for current consumption in period 1 (hence the subscript) if he has a in assets at the start of that period
  - If $a = A_1$ then it follows readily that $C_1^* = \mathbf{C}_1(A_1;\cdot)$
  - If $a < A_1$ the conditionally optimal solution is feasible but suboptimal
  - If $a > A_1$ the conditionally optimal solution is infeasible as the consumer does not possess that much in financial assets
- The *policy function* $\mathbf{A}_1^+(a;\cdot)$ in (S9b) represents the conditionally optimal choice that the agents makes in period 1 (hence the subscript) about the level of assets he want to carry over to the next period (hence the superscript '$+$ ')
  - Obviously $A_2^* = \mathbf{A}_1^+(a;\cdot)$ for $a = A_1$ only

Alternative viewpoint (4)

- Now consider the consumer with a in assets in period $t = 2$. What does he choose for C_2 and A_3?
- The answer is obtained by maximizing $\ln C_2 + \beta \ln C_3$ subject to:

$$(1 + r_2)a + w_2 + \frac{w_3}{1 + r_3} = C_2 + \frac{C_3}{1 + r_3}$$

- This gives the policy functions:

$$\begin{aligned}\hat{C}_2 &= \mathbf{C}_2(a; r_2, r_3, w_2, w_3) \\ &\equiv \frac{1}{1+\beta}\left[(1 + r_2)a + w_2 + \frac{w_3}{1 + r_3}\right] \qquad \text{(S9c)} \\ \hat{A}_3 &= (1 + r_2)a + w_2 - \hat{C}_1 = \mathbf{A}_2^{+}(a; r_2, r_3, w_2, w_3) \\ &\equiv \frac{\beta}{1+\beta}\left[(1 + r_2)a + w_2\right] - \frac{1}{1+\beta}\frac{w_3}{1 + r_3} \qquad \text{(S9d)}\end{aligned}$$

Alternative viewpoint (5)

- Obviously, $\hat{C}_2 = C_2^*$ and $\hat{A}_3 = A_3^*$ if and only if $a = A_2^*$, i.e. $C_2^* = \mathbf{C}_2(A_2^*;\cdot)$ and $A_3^* = \mathbf{A}_2^+(A_2^*;\cdot)$
- Finally, consider the consumer who has a in assets in period $t = 3$. What does he choose for C_3 and A_4?
- The answer is obtained by maximizing $\ln C_3$ subject to $(1 + r_3)A_3 + w_3 = C_3 + A_4$ and $A_4 \geq 0$
- This gives the policy functions:

$$\hat{C}_3 = \mathbf{C}_3(a; r_3, w_3) \equiv (1 + r_3)a + w_3 \tag{S9e}$$

$$\hat{A}_4 = \mathbf{A}_3^+(a; r_3, w_3) \equiv 0 \tag{S9f}$$

- Just as before, $\hat{C}_3 = C_3^*$ if and only if $a = A_3^*$, i.e. so that $C_3^* = \mathbf{C}_3(A_3^*;\cdot)$
- Unlike what we found before, however, $\hat{A}_4 = A_4^* = 0$ regardless of a, i.e. the consumer will always deplete resources completely in the final period of life

DP insight (1)

- PRINCIPLE OF OPTIMALITY. An optimal policy has the property that whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (Bellman, 1957, p. 83)
- DP solves a complex multi-stage problem by breaking it up into a number of smaller subproblems
- DP computes *value functions* which depend on the state variable at each time
- Return to the decision problem of the consumer who lives for three periods
- We start at the end of life and work back to the first period

DP: Choice problem in period $t = 3$

- The objective is to maximize $V_3 \equiv U(C_3)$, subject to the budget constraint, $A_4 = (1+r_3)A_3 + w_3 - C_3$
- Consumer with a in assets chooses $\hat{C}_3 = \mathbf{C}_3(a; r_3, w_3)$ and $\hat{A}_4 = \mathbf{A}_3^+(a; r_3, w_3) \equiv 0$
- Substituting $\hat{C}_3$ into the felicity function gives the value function for period $t = 3$ in terms of a:

$$V_3(a) \equiv U(\mathbf{C}_3(a; r_3, w_3)) = \ln\left[(1+r_3)a + w_3\right] \qquad \text{(S10a)}$$

- Note its derivative:

$$V_3'(a) = (1+r_3)U'(\mathbf{C}_3(a; r_3, w_3)) = \frac{1+r_3}{\mathbf{C}_3(a; r_3, w_3)} \qquad \text{(S10b)}$$

DP: Choice problem in period $t = 2$

- The objective is to maximize $V_2 \equiv U(C_2) + \beta U(C_3)$, subject to the budget constraint, $A_3 = (1 + r_2)A_2 + w_2 - C_2$
- Choice of current consumption c and the amount of assets to carry over into the third period a^+
- Choice of a^+ will ensure that the value function in period 3 is equal to $V_3(a^+)$
- Choice problem in period 2:

$$V_2(a) = \max_{c,a^+} \quad U(c) + \beta V_3(a^+)$$

$$\text{subject to:} \quad a^+ = (1 + r_2)a + w_2 - c \qquad \text{(S10c)}$$

- Eq. (S10c) is the *Bellman Equation*
- FONC for the maximization problem on the right-hand side:

$$U'(c) = \beta V_3'((1 + r_2)a + w_2 - c) \qquad \text{(S10d)}$$

DP: Choice problem in period $t = 2$

- This is an implicit relationship between c and a, the solution of which gives the *policy function* $\mathbf{C}_2(a, r_2, w_2)$
- For the logarithmic felicity function $U(c) = \ln c$ equation (S10d) simplifies to:

$$\frac{1}{c} = \frac{\beta(1+r_3)}{(1+r_3)[(1+r_2)a + w_2 - c] + w_3} \qquad \text{(S10e)}$$

- Solving for $c = \mathbf{C}_2(a; \cdot)$ we find:

$$\mathbf{C}_2(a; \cdot) \equiv \frac{1}{1+\beta}\left[(1+r_2)a + w_2 + \frac{w_3}{1+r_3}\right] \qquad \text{(S10f)}$$

DP: Choice problem in period $t = 2$

- The *policy function* for a^+ follows from the constraint, $\mathbf{A}_2^+(a, r_2, w_2) = (1+r_2)a + w_2 - \mathbf{C}_2(a, r_2, w_2)$:

$$\mathbf{A}_2^+(a;\cdot) = \frac{\beta}{1+\beta}\left[(1+r_2)a + w_2\right] - \frac{1}{1+\beta}\frac{w_3}{1+r_3} \quad \text{(S10g)}$$

- By substituting $\mathbf{C}_2(a;\cdot)$ and $\mathbf{A}_2^+(a;\cdot)$ into (S10c) we find the value function for period $t = 2$:

$$\begin{aligned} V_2(a) &\equiv U(\mathbf{C}_2(a;\cdot)) + \beta V_3(\mathbf{A}_2^+(a;\cdot)) \\ &= \ln\left(\frac{\beta^\beta}{(1+\beta)^{1+\beta}}\right) + \beta\ln(1+r_3) \\ &\quad + (1+\beta)\ln\left[(1+r_2)a + w_2 + \frac{w_3}{1+r_3}\right] \end{aligned} \quad \text{(S10h)}$$

Digression: Benveniste-Scheinkman Theorem

- In (S10d) we find an intimate relationship between the derivative of the value function, $V_3'(a)$, and marginal utility, $U'(\mathbf{C}_3(a;\cdot))$
- This is not a coincidental result
- To show the result for $t = 2$ we use (S10h) to find:

$$
\begin{aligned}
V_2'(a) &\equiv U'(\mathbf{C}_2(a;\cdot))\frac{d\mathbf{C}_2(a;\cdot)}{da} + \beta V_3'(\mathbf{A}_2^+(a;\cdot))\Big[(1+r_2) \\
&\quad - \frac{d\mathbf{C}_2(a;\cdot)}{da}\Big] \\
&= \left[U'(\mathbf{C}_2(a;\cdot)) - \beta V_3'(\mathbf{A}_2^+(a;\cdot))\right]\frac{d\mathbf{C}_2(a;\cdot)}{da} \\
&\quad + \beta(1+r_2)V_3'(\mathbf{A}_2^+(a;\cdot)) \\
&= (1+r_2)U'(\mathbf{C}_2(a;\cdot)) \qquad \text{(S10i)}
\end{aligned}
$$

DP: Choice problem in period $t = 1$

- The problem:

$$V_1(a) = \max_{c,a^+} \quad U(c) + \beta V_2(a^+)$$
$$\text{subject to:} \quad a^+ = (1+r_1)a + w_1 - c \qquad \text{(S10j)}$$

- FONC:

$$U'(c) = \beta V_2'((1+r_1)a + w_1 - c)) \qquad \text{(S10k)}$$

- For $U(c) = \ln c$ to:

$$\frac{1}{c} = \frac{\beta(1+\beta)}{(1+r_1)a + w_1 + \frac{w_2}{1+r_2} + \frac{w_3}{(1+r_2)(1+r_3)}} \qquad \text{(S10l)}$$

DP: Choice problem in period $t = 1$

- Policy function for consumption:

$$\mathbf{C}_1(a;\cdot) \equiv \frac{1}{1+\beta+\beta^2}\Bigg[(1+r_1)a + w_1 + \frac{w_2}{1+r_2} + \frac{w_3}{(1+r_2)(1+r_3)}\Bigg] \tag{S10m}$$

- Policy function for future assets:

$$\mathbf{A}_1^+(a;\cdot) = \frac{\beta(1+\beta)}{1+\beta+\beta^2}\left[(1+r_1)a + w_1\right] - \frac{1}{1+\beta+\beta^2}\left[\frac{w_2}{1+r_2} + \frac{w_3}{(1+r_2)(1+r_3)}\right] \tag{S10n}$$

DP: Choice problem in period $t = 1$

- Value function:

$$
\begin{aligned}
V_1(a) &= U(\mathbf{C}_1(a;\cdot) + \beta V_2(\mathbf{A}_1^{+}(a;\cdot)) \\
&= -(1+\beta+\beta^2)\ln(1+\beta+\beta^2) + \beta(1+2\beta)\ln\beta \\
&\quad + (1+\beta(1+\beta))\ln\left[(1+r_1)a + w_1 + \frac{w_2}{1+r_2}\right. \\
&\quad \left. + \frac{w_3}{(1+r_2)(1+r_3)}\right] \\
&\quad + \beta(1+\beta)\ln(1+r_2) + \beta^2\ln(1+r_3) \qquad \text{(S10o)}
\end{aligned}
$$

Summary of the DP method (1)

- We have computed the policy functions for consumption, $\mathbf{C}_t(a;\cdot)$, for future assets, $\mathbf{A}_t^+(a;\cdot)$, and the value functions, $V_t(a)$, all in terms of a
- Application: consider an agent who has A_1 in assets
- Maximum attainable (lifetime) utility level is $V_1(A_1)$
- Optimal consumption in the first period is $C_1^* = \mathbf{C}_1(A_1;\cdot)$
- Optimal assets at the start of the second period is $A_2^* = \mathbf{A}_1^+(A_1;\cdot)$
- In the second period we find that $C_2^* = \mathbf{C}_2(A_2^*;\cdot)$ and $A_3^* = \mathbf{A}_2^+(A_2^*;\cdot)$
- In the third period we find $C_3^* = \mathbf{C}_3(A_3^*;\cdot)$ and $A_4^* = \mathbf{A}_3^+(A_3^*;\cdot) = 0$

Summary of the DP method (2)

- We visualize the value functions $V_t(a)$ as well as the policy functions for consumption $\mathbf{C}_t(a)$ and next-period's financial assets $\mathbf{A}_t^+(a)$ in **Figure 17.1**
- Schematically the method of dynamic programming in a T-period finite horizon setting thus proceeds as follows
- Compute the value function for the final period, $V_T(a)$, as well as the policy functions, $\mathbf{C}_T(a)$ and $\mathbf{A}_T^+(a)$
- Use the Bellman equation to compute $V_{T-1}(a)$ and the policy functions $\mathbf{C}_{T-1}(a)$ and $\mathbf{A}_{T-1}^+(a)$. Continue this step until $V_1(a)$, $\mathbf{C}_1(a)$ and $\mathbf{A}_1^+(a)$ are obtained
- Impose the initial condition, $a = A_1$ and iterate forward in time to compute the optimal choices, $C_1^* = \mathbf{C}_1(A_1)$, $A_2^* = \mathbf{A}_1^+(A_1)$, $C_2^* = \mathbf{C}_2(A_2^*)$, $A_3^* = \mathbf{A}_2^+(A_2^*)$, $\ldots$, $C_T^* = \mathbf{C}_T(A_T^*)$, $A_{T+1}^* = \mathbf{A}_T^+(A_T^*) = 0$

Figure 17.1: Value functions and policy functions

(a) Value functions: $V_t(a)$

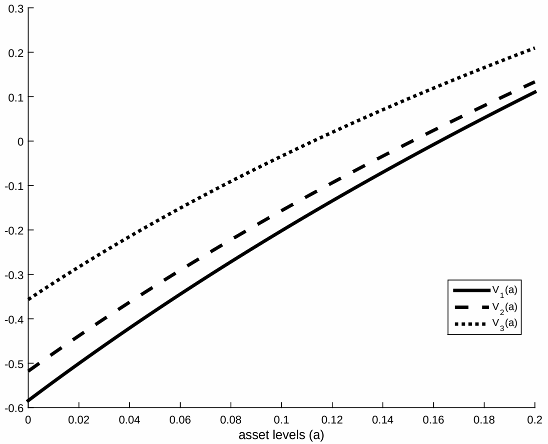


Figure 17.1: Value functions and policy functions

(b) Policy functions for consumption: $\mathbf{C}_t(a)$

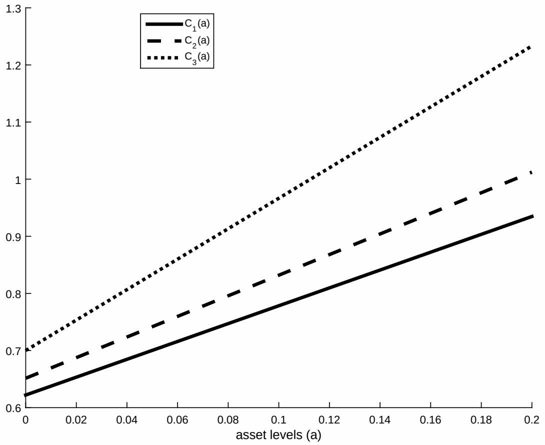
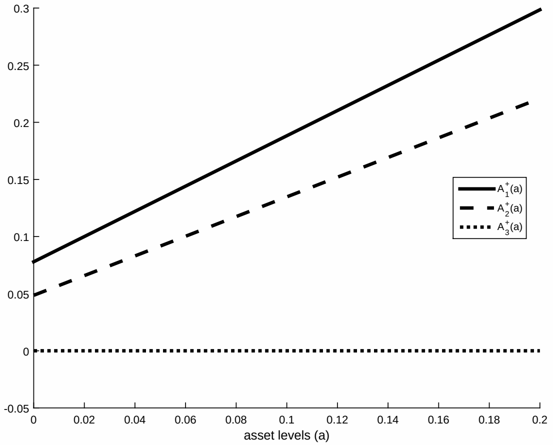


Figure 17.1: Value functions and policy functions

(c) Policy functions for next period's assets: $\mathbf{A}_t^+(a)$



DP: Infinite planning problem (1)

- Lifetime utility:

$$\Lambda_1 \equiv \sum_{t=1}^{\infty} \beta^{t-1} U(C_t) \tag{S11a}$$

- Financial asset accumulation:

$$A_{t+1} = (1+r_t)A_t + w_t - C_t \tag{S11b}$$

- Use the method of DP right from the start
- There is no final period so we cannot start by computing $V_T(a)$
- Instead we postulate the Bellman equation for period t as:

$$V_t(a) = \max_{c,a^+} \quad U(c) + \beta V_{t+1}(a^+)$$

$$\text{subject to:} \quad a^+ = (1+r_t)a + w_t - c \tag{S11c}$$

DP: Infinite planning problem (2)

- The first-order condition for c is:

$$U'(c) = \beta V'_{t+1}((1+r_t)a + w_t - c) \tag{S11c}$$

- In principle we could solve (S11c) for $c = \mathbf{C}_t(a)$, $\mathbf{A}_t^+(a) = (1+r_t)a + w_t - \mathbf{C}_t(a)$, and find:

$$V_t(a) = U(\mathbf{C}_t(a)) + \beta V_{t+1}(\mathbf{A}_t^+(a)) \tag{S11d}$$

- Oops, we do not know the functional form of $V'_{t+1}(a^+)$ so this seems to be a dead end!
- But the Benveniste-Scheinkman Theorem furnishes a link between the derivative of the value function and marginal utility

DP: Infinite planning problem (3)

- By differentiating (S11d) with respect to a we find:

$$\begin{aligned} V_t'(a) &= U'(\mathbf{C}_t(a))\frac{d\mathbf{C}_t(a)}{da} + \beta V_{t+1}'(\mathbf{A}_t^+(a))\left[(1+r_t) - \frac{d\mathbf{C}_t(a)}{da}\right] \\ &= \left[U'(\mathbf{C}_t(a)) - \beta V_{t+1}'(\mathbf{A}_t^+(a))\right]\frac{d\mathbf{C}_t(a)}{da} \\ &\quad + \beta(1+r_t)V_{t+1}'(\mathbf{A}_t^+(a)) \\ &= (1+r_t)U'(\mathbf{C}_t(a)) \qquad \text{(S11e)} \end{aligned}$$

- By induction we find that:

$$V_{t+1}'(a^+) = (1+r_{t+1})U'(c^+) \qquad \text{(S11f)}$$

DP: Infinite planning problem (4)

- Hence, (S11c) can be rewritten as:

$$U'(c) = \beta(1 + r_{t+1})U'(c^{+}) \tag{S11g}$$

- Put differently, consumption in adjacent periods will be related according to the usual Euler equation:

$$U'(C_t) = \beta(1 + r_{t+1})U'(C_{t+1}) \tag{S11h}$$

which simplifies for the logarithmic felicity function to:

$$\frac{C_{t+1}}{C_t} = \beta(1 + r_{t+1}) \tag{S11i}$$

- The Euler equation is a vital piece of information which often allows us to compute the optimal solutions for current and future consumption without any further need for value functions or policy functions

Stochastic 3-period example (1)

- Back to the three-period consumption-savings model
- Agent faces idiosyncratic labour productivity risk
- Expected utility at birth:

$$E_1[\Lambda_1] = U(C_1) + \beta E_1[U(C_2)] + \beta^2 E_1[U(C_3)] \qquad \text{(S12a)}$$

- Financial assets accumulation:

$$A_{t+1} = (1 + r_t)A_t + \eta_t w_t - C_t \qquad \text{(S12b)}$$

  - η_t is a stochastic variable representing labour productivity risk (nature draws realizations over time)
  - r_t and w_t are taken as given

- Terminal constraint of the form:

$$A_{t+4} \geq 0 \qquad \text{(S12c)}$$

Stochastic 3-period example (2)

- Simple stochastic process for η_t:
  1. At birth the agent has an average productivity level, $\eta_1 = e_2 = 1$
  2. For later periods, η_t follows a three state stationary Markov Scheme, i.e. $\eta_t \in \{e_1, e_2, e_3\}$
  3. The transition probabilities are defined as:

$$p_{ij} = \mathrm{Prob}(\eta_{t+1} = e_j | \eta_t = e_i) \tag{S12d}$$

     For obvious reasons it must be the case that $\sum_{j=1}^{3} p_{ij} = 1$
  4. The transition matrix is defined as:

$$P \equiv \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \tag{S12e}$$

     We assume that $0 < p_{ij} < 1$ so there are no absorbing states. Since $p_{13}, p_{13} > 0$ spectacular reversals of fortune are possible.

Figure 17.2: Markov process for labour productivity

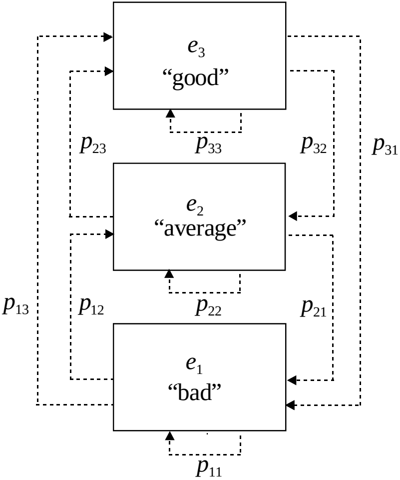
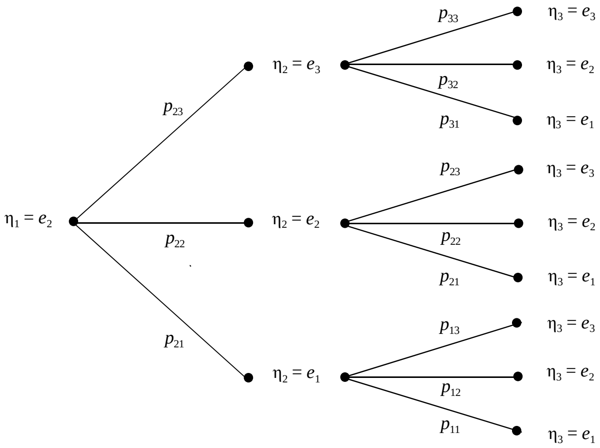


Figure 17.3: Labour productivity over the life cycle



Stochastic 3-period example (3)

- Since $\eta_1 = e_2$ by assumption the initial unconditional probability distribution of η_1 is trivial:

$$\pi_1 \equiv \begin{bmatrix} \pi_{11} \\ \pi_{12} \\ \pi_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{S12f}$$

- To obtain the next period's unconditional probability distribution of η_2 we use the result that $\pi_2' = \pi_1' P$, where P is given in (S12e) above:

$$\pi_2 \equiv \begin{bmatrix} \pi_{21} \\ \pi_{22} \\ \pi_{23} \end{bmatrix} = \begin{bmatrix} p_{21} \\ p_{22} \\ 1 - p_{21} - p_{22} \end{bmatrix} \tag{S12g}$$

  From the perspective of period $t = 1$ ('unconditionally') the consumer assigns probability π_{2j} to being in state j in period $t = 2$

Stochastic 3-period example (4)

- The unconditional probability distribution of η_3 is determined by $\pi_3' = \pi_2' P$:

$$\pi_3 \equiv \begin{bmatrix} \pi_{31} \\ \pi_{32} \\ \pi_{33} \end{bmatrix} = \begin{bmatrix} p_{21}(p_{11} + p_{22}) + p_{23}p_{31} \\ p_{12}p_{21} + p_{22}^2 + p_{23}p_{32} \\ p_{13}p_{21} + p_{23}\left(p_{22} + p_{33}\right) \end{bmatrix} \qquad \text{(S12h)}$$

Traditional “brute force” method (1)

- What is the optimal consumption level in the first period?
- Expected utility in terms of asset levels:

$$\begin{aligned}E_1[\Lambda_1] = {} & U((1+r_1)A_1 + e_2 w_1 - A_2) \\ & + \beta\Big[\pi_{21}U((1+r_2)A_2 + e_1 w_2 - A_3) \\ & \quad + \pi_{22}U((1+r_2)A_2 + e_2 w_2 - A_3) \\ & \quad + \pi_{23}U((1+r_2)A_2 + e_3 w_2 - A_3)\Big] \\ & + \beta^2\Big[\pi_{31}U((1+r_3)A_3 + e_1 w_3 - A_4) \\ & \quad + \pi_{32}U((1+r_3)A_3 + e_2 w_3 - A_4) \\ & \quad + \pi_{33}U((1+r_3)A_3 + e_3 w_3 - A_4)\Big] \qquad \text{(S13a)}\end{aligned}$$

- Choice variables are A_2, A_3, and A_4

Traditional "brute force" method (2)

- No unused assets are planned (Kuhn-Tucker conditions):

$$\begin{aligned}\frac{\partial E_1[\Lambda_1]}{\partial A_4} = -\beta^2 \Big[&\pi_{31} U'((1+r_3)A_3 + e_1 w_3 - A_4) \\ &+ \pi_{32} U'((1+r_3)A_3 + e_2 w_3 - A_4) \\ &+ \pi_{33} U'((1+r_3)A_3 + e_3 w_3 - A_4) \Big] < 0,\end{aligned}$$

$$A_4 \geq 0, \qquad A_4 \frac{\partial E_1[\Lambda_1]}{\partial A_4} = 0 \tag{S13b}$$

It follows that $A_4^* = 0$ just as in the deterministic case

Traditional "brute force" method (3)

- Assets in period $t = 2$:

$$\begin{aligned}\frac{\partial E_1[\Lambda_1]}{\partial A_2} &= -U'((1+r_1)A_1 + w_1 - A_2) \\ &\quad + \beta(1+r_2)\Big[\pi_{21}U'((1+r_2)A_2 + e_1 w_2 - A_3) \\ &\qquad + \pi_{22}U'((1+r_2)A_2 + e_2 w_2 - A_3) \\ &\qquad + \pi_{23}U'((1+r_2)A_2 + e_3 w_2 - A_3)\Big] = 0 \end{aligned} \tag{S13c}$$

Traditional "brute force" method (4)

- Assets in period $t = 3$:

$$\begin{aligned}\frac{\partial E_1[\Lambda_1]}{\partial A_3} = -\beta\Big[&\pi_{21}U'((1+r_2)A_2+e_1w_2-A_3)\\ &+\pi_{22}U'((1+r_2)A_2+e_2w_2-A_3)\\ &+\pi_{23}U'((1+r_2)A_2+e_3w_2-A_3)\Big]\\ +\beta^2(1+r_3)\Big[&\pi_{31}U'((1+r_3)A_3+e_1w_3)\\ &+\pi_{32}U'((1+r_3)A_3+e_2w_3)\\ &+\pi_{33}U'((1+r_3)A_3+e_3w_3)\Big]=0 \qquad \text{(S13d)}\end{aligned}$$

Traditional "brute force" method (5)

- Eqns. (S13c)-(S13d) represent two equations in two unknowns which can in **principle** be solved for A_2^* and A_3^* in terms of the parameters of the problem (i.e., r_2, r_3, w_2, w_3, e_j, and p_{ij})
- Even for $U(c) \equiv \ln c$ there are **no analytical solutions**
- **Numerical methods** must be used to find A_2^* and A_3^*
- This gives us two points on the policy functions:

$$A_2^* = \mathbf{A}_1^+(A_1, e_2) \tag{S13e}$$

$$C_1^* = \mathbf{C}_1(A_1, e_2) = (1 + r_1)A_1 + e_2 w_1 - \mathbf{A}_1^+(A_1, e_2) \tag{S13f}$$

Traditional "brute force" method (6)

- In period $t = 2$, the consumer has $A_2^* = \mathbf{A}_1^+(A_1, e_2)$ for sure but he enters the risky part of life as nature reveals the realization of η_2
- If the agent gets $\eta_2 = e_i$ then expected utility from the perspective of period $t = 2$ is given by:

$$E_2[\Lambda_2(A_2^*, e_i)] = U((1+r_2)A_2^* + e_i w_2 - A_3) + \beta \sum_{j=1}^{3} p_{ij} U((1+r_3)A_3 + e_j w_3) \quad \text{(S13g)}$$

- The only choice variable is A_3

Traditional "brute force" method (7)

- FONC:

$$\frac{dE_2[\Lambda_2(A_2^*, e_i)]}{dA_3} = -U'((1+r_2)A_2^* + e_i w_2 - A_3) + \beta(1+r_3)\sum_{j=1}^{3} p_{ij} U'((1+r_3)A_3 + e_j w_3) = 0 \tag{S13h}$$

- Numerically, equation (S13h) can easily be solved for A_3^*
- This gives us the points on the policy functions if the state is (A_2^*, e_i):

$$A_3^* = \mathbf{A}_2^+(A_2^*, e_i) \tag{S13i}$$

$$C_2^* = \mathbf{C}_2(A_2^*, e_i) = (1+r_2)A_2 + e_i w_2 - \mathbf{A}_2^+(A_2, e_i) \tag{S13j}$$

Traditional "brute force" method (8)

- In period $t = 3$, the consumer has $A_3^* = \mathbf{A}_2^+(A_2, e_i)$ in financial assets and the value of $\eta_3 = e_j$ is revealed
- The optimal choices are trivial:

$$C_3^* = \mathbf{C}_3(A_3^*, e_j) = (1 + r_3)A_3^* + e_j w_3 \tag{S13k}$$

$$A_4^* = \mathbf{A}_3^+(A_3^*, e_j) = 0 \tag{S13l}$$

- **Conclusion**
  - It is feasible though tedious to compute the optimal choices in the traditional manner by repeatedly solving a maximization problem involving expected remaining lifetime utility
  - With idiosyncratic labour productivity risk of the Markov form, the state vector in a particular period consists of assets at the start of the period as well as the productivity indicator for that period

DP: Choice problem in period $t = 3$

- Start at the end of life and work back to the first period
- The consumer who has a in assets and labour productivity η will choose c and a^+ in order to maximize $U(c) = \ln c$ subject to $a^+ = (1 + r_3)a + \eta w_3 - c$
- Policy and value functions:

$$\mathbf{C}_3(a, \eta; r_3, w_3) \equiv (1 + r_3)a + \eta w_3 \tag{S14a}$$

$$\mathbf{A}_3^+(a, \eta; r_3, w_3) = 0 \tag{S14b}$$

$$\begin{aligned} V_3(a, \eta) &\equiv U(\mathbf{C}_3(a, \eta; r_3, w_3)) \\ &= \ln\left[(1 + r_3)a + \eta w_3\right] \end{aligned} \tag{S14c}$$

DP: Choice problem in period $t = 3$

- Note that:

$$V_3'(a, \eta) \equiv \frac{\partial V_3(a, \eta)}{\partial a} = (1 + r_3)U'(\mathbf{C}_3(a, \eta; \cdot)) = \frac{1 + r_3}{(1 + r_3)a + \eta w_3} \tag{S14d}$$

- Since η has three possible realizations, there are three each of the $V_3(a, \eta)$ and $V_3'(a, \eta)$ functions that must be computed

DP: Choice problem in period $t = 2$

- Choice problem in period $t = 2$
- Bellman equation:

$$V_2(a, \eta) = \max_{c, a^+} \quad U(c) + \beta E_{\eta^+|\eta}\left[V_3(a^+, \eta^+)\right]$$

$$\text{subject to:} \quad a^+ = (1 + r_2)a + \eta w_2 - c \qquad \text{(S14e)}$$

- Here $E_{\eta^+|\eta}\left[\cdot\right]$ stands for the conditional expectations operator so that:

$$E_{\eta^+|\eta}\left[V_3(a^+, \eta^+)\right] = \sum_{j=1}^{3} p_{ij} V_3(a^+, \eta_j) \qquad \text{(S14f)}$$

where we let $\eta = e_i$ and $\eta^+ = e_j$

DP: Choice problem in period $t = 2$

- First-order condition for c:

$$U'(c) = \beta E_{\eta^+|\eta}\left[V_3'(a^+, \eta^+)\right] = \beta \sum_{j=1}^{3} p_{ij} V_3'(a^+, \eta_j) \quad \text{(S14g)}$$

- For $U(c) = \ln c$, $\eta = e_i$, and $\eta^+ = e_j$ we find that (S14g) reduces to:

$$\frac{1}{c} = \beta(1+r_3) \sum_{j=1}^{3} \frac{p_{ij}}{(1+r_3)a^+ + e_j w_3}$$

$$= \beta \sum_{j=1}^{3} \frac{p_{ij}}{(1+r_2)a + e_i w_2 - c + e_j \frac{w_3}{1+r_3}}$$

- Numerical methods give us $c = \mathbf{C}_2(a, \eta; \cdot)$ and $\mathbf{A}_2^+(a, \eta; \cdot) \equiv (1+r_2)a + \eta w_2 - \mathbf{C}_2(a, \eta; \cdot)$

DP: Choice problem in period $t = 2$

- The value function in the second period is given by:

$$V_2(a,\eta) = U(\mathbf{C}_2(a,\eta;\cdot)) + \beta E_{\eta^+|\eta}[V_3(\mathbf{A}_2^+(a,\eta;\cdot),\eta^+)] \tag{S14h}$$

- Differentiate with respect to a to obtain the Benveniste-Scheinkman result in a stochastic setting:

$$\begin{aligned} V_2'(a,\eta) &\equiv U'(\mathbf{C}_2(a,\eta;\cdot))\frac{d\mathbf{C}_2(a,\eta;\cdot)}{da} \\ &+ \beta E_{\eta^+|\eta}\left[V_3'(\mathbf{A}_2^+(a,\eta;\cdot),\eta^+)\left((1+r_2) - \frac{d\mathbf{C}_2(a,\eta;\cdot)}{da}\right)\right] \\ &= (1+r_2)U'(\mathbf{C}_2(a,\eta;\cdot)) \end{aligned} \tag{S14i}$$

DP: Choice problem in period $t = 1$

- Bellman equation:

$$V_1(a, \eta) = \max_{c, a^+} \quad U(c) + \beta E_{\eta^+|\eta} \left[V_2(a^+, \eta^+) \right]$$

$$\text{subject to:} \quad a^+ = (1 + r_1)a + \eta w_1 - c, \tag{S14j}$$

- First-order condition for c:

$$U'(c) = \beta E_{\eta^+|\eta} \left[V_2'(a^+, \eta^+) \right] \tag{S14k}$$

- The policy functions are $\mathbf{C}_1(a, \eta; \cdot)$ and $\mathbf{A}_1^+(a, \eta; \cdot)$, and the value function is:

$$V_1(a, \eta) = U(\mathbf{C}_1(a, \eta; \cdot)) + \beta E_{\eta^+|\eta} \left[V_2(\mathbf{A}_2^+(a, \eta; \cdot), \eta^+) \right] \tag{S14l}$$

Stochastic DP: Closing remarks

- Stochastic consumption Euler equations:

$$
\begin{aligned}
U'(\mathbf{C}_2(a,\eta;\cdot)) &= \beta(1+r_3)E_{\eta^+|\eta}\left[U'(\mathbf{C}_3(a^+,\eta^+;\cdot))\right] \\
U'(\mathbf{C}_1(a,\eta;\cdot)) &= \beta(1+r_2)E_{\eta^+|\eta}\left[U'(\mathbf{C}_2(a^+,\eta^+;\cdot))\right]
\end{aligned}
$$

- Not quite as useful here as they are in a deterministic setting
- To solve the consumer's choice problem we must compute the policy functions and value functions numerically
- **Figure 17.4** plots value functions
- **Figure 17.5** plots policy functions
- **Table 17.1** gives some numbers
- Model can explain consumption and wealth inequality

Figure 17.3: Value functions – stochastic case

, (a) Value function for period $t = 1$: $V_1(a, e_2)$

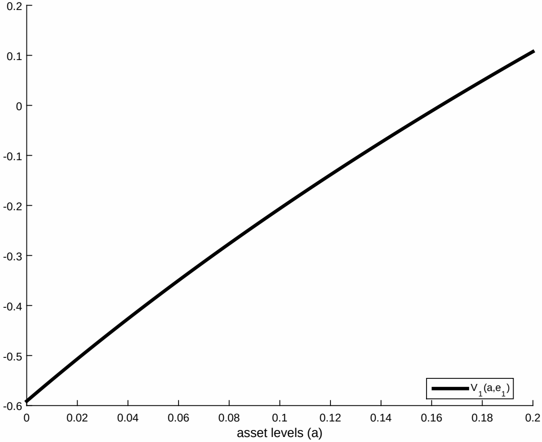


Figure 17.3: Value functions – stochastic case

(b) Value functions for period $t = 2$: $V_2(a, \eta)$

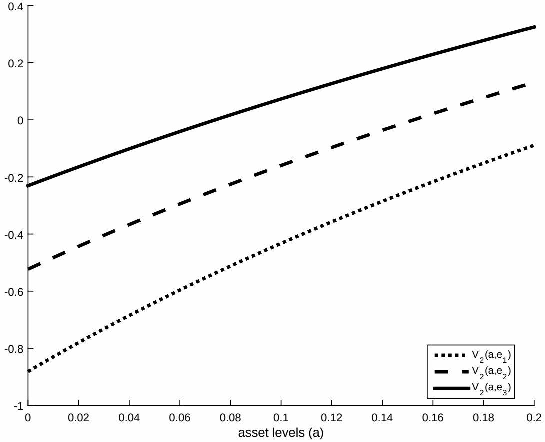


Figure 17.3: Value functions – stochastic case

(c) Value functions for period $t = 3$: $V_3(a, \eta)$

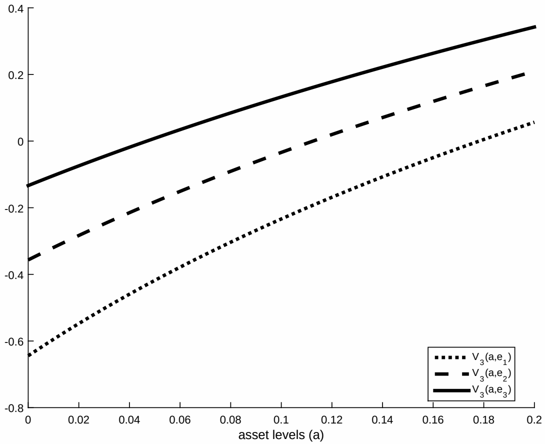


Figure 17.4: Policy functions – stochastic case

(a) Assets: $\mathbf{A}_1^+(a, e_2)$

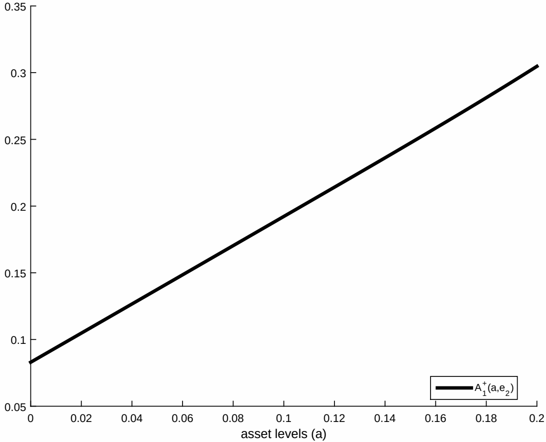


Figure 17.4: Policy functions – stochastic case

(b) Consumption: $\mathbf{C}_2(a, \eta)$

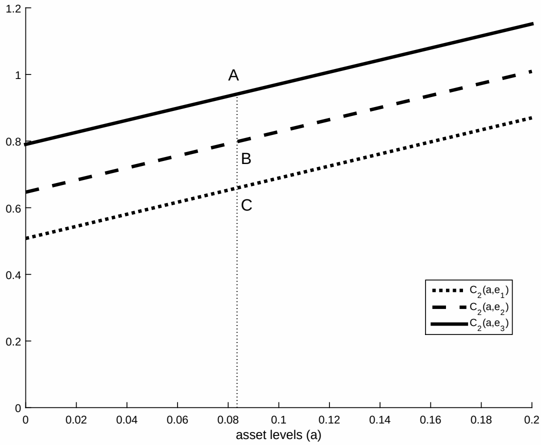


Figure 17.4: Policy functions – stochastic case

(c) Assets: $\mathbf{A}_2^+(a, \eta)$

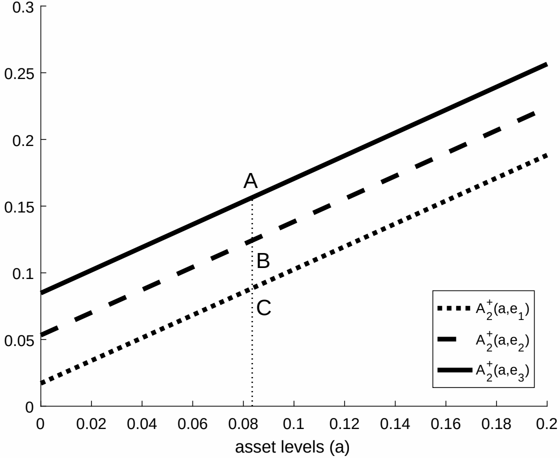


Figure 17.4: Policy functions – stochastic case

(d) Consumption: $\mathbf{C}_3(a, \eta)$

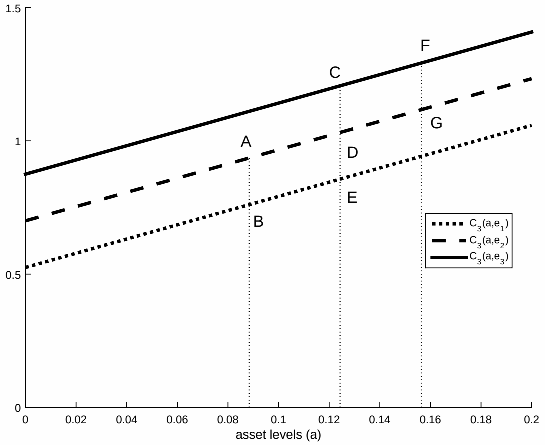


Table 17.1: Some numerical examples

(b) **Sequential stochastic choices**

Choices made in period 1:		
Consumption:	C_1^*	0.6165
Assets:	A_2^*	0.0835
Choices made in period 2:		
Consumption:	$C_2^*(e_1)$	0.6591
	$C_2^*(e_2)$	0.7982
	$C_2^*(e_3)$	0.9411
Assets:	$A_3^*(e_1)$	0.0884
	$A_3^*(e_2)$	0.1243
	$A_3^*(e_3)$	0.1564

Table 17.1: Some numerical examples

(b) **Sequential stochastic choices**

Choices made in period 3:		
Consumption:	$C_3^*(e_1, e_1)$	0.7607
	$C_3^*(e_1, e_2)$	0.9357
	$C_3^*(e_2, e_1)$	0.8564
	$C_3^*(e_2, e_2)$	1.0314
	$C_3^*(e_2, e_3)$	1.2064
	$C_3^*(e_3, e_2)$	1.1171
	$C_3^*(e_3, e_3)$	1.2921
Assets:	$A_4^*(e_i, e_j)$	0.0000

Optimal stochastic growth model (1)

- Focus on the social planning solution
  - Large population of identical consumers
  - Population size is constant and normalized to unity
  - The benevolent social planner maximizes the utility function of the representative consumer
- Objective function:

$$\Omega_0 \equiv E_0 \left[\sum_{t=0}^{\infty} \beta^t U(C_t) \right] \tag{S15a}$$

- Macroeconomic resource constraint:

$$C_t + K_{t+1} = Z_t F(K_t, 1) + (1 - \delta) K_t \tag{S15b}$$

where Z_t is the random technology shock

Optimal stochastic growth model (1)

- The history of all technology shocks that have occurred at or before time t by h^t:

$$h^t \equiv (Z_0, Z_1, \ldots, Z_t) \tag{S15c}$$

- In the most general case the optimal plans that the social planner formulates at time t will depend on the entire vector h^t
- If we assume that the stochastic process has the Markov property, however, then Z_t is all the planner needs to know to make optimal plans at time t
- Two constraints, (a) consumption must be non-negative $C_t \geq 0$ and (b) at time $t = 0$ the existing capital stock is given (K_0 is fixed) and Z_0 is known

Traditional method (1)

- The traditional approach to solve the social planning problem exploits the Markov assumption for the technology shocks and writes the Lagrangian at time $t = 0$ as:

$$\begin{aligned}\mathcal{L}_0 = {} & U(Z_0 F(K_0, 1) + (1-\delta)K_0 - K_1) \\ & + \beta E_{Z_1|Z_0}\left[U(Z_1 F(K_1, 1) + (1-\delta)K_1 - K_2)\right] \\ & + \beta^2 E_{Z_2|Z_0}\left[U(Z_2 F(K_2, 1) + (1-\delta)K_2 - K_3)\right] + \ldots\end{aligned}$$

  where the expectation operator $E_{Z_{t+1}|Z_t}\left[\phi(Z_{t+1})\right]$ stands for the conditional expectation of $\phi(Z_{t+1})$ given Z_t
- Since K_0 is predetermined, the only choice that is made and executed at time $t = 0$ is the one about K_1

Traditional method (2)

- First/order condition:

$$\frac{\partial \mathcal{L}_0}{\partial K_1} = -U'(Z_0 F(K_0, 1) + (1-\delta)K_0 - K_1) + \beta E_{Z_1|Z_0}\Big[U'(Z_1 F(K_1, 1) + (1-\delta)K_1 - K_2) \times \Big(Z_1 F_K(K_1, 1) + 1 - \delta\Big)\Big] = 0$$

where $F_K(K_1, 1)$ is the marginal product of capital when the stock equals K_1

- Rewrite:

$$U'(C_0) = \beta E_{Z_1|Z_0}\Big[U'(C_1) \cdot \Big(Z_1 F_K(K_1, 1) + 1 - \delta\Big)\Big] \quad \text{(S15d)}$$

Traditional method (2)

- Define the "implicit real interest rate" as:

$$r_{t+1} \equiv Z_{t+1} F_K(K_{t+1}, 1) - \delta \qquad \text{(S15e)}$$

- For any arbitrary period t the social optimum is characterized by the corresponding stochastic Euler equation:

$$U'(C_t) = \beta E_{Z_{t+1}|Z_t} \left[U'(C_{t+1}) \cdot (1 + r_{t+1}) \right] \qquad \text{(S15f)}$$

DP: Choice problem in period t

- At time t the control variable is consumption for that time period, C_t, whilst the state variables are the capital stock and the technology indicator, K_t and Z_t
- Many writers express the Bellman equation as:

$$V(K_t, Z_t) = \max_{C_t, K_{t+1}} U(C_t) + \beta E_{Z_{t+1}|Z_t} \left[V(K_{t+1}, Z_{t+1})\right]$$

$$\text{subject to } K_{t+1} = Z_t F(K_t, 1) + (1-\delta)K_t - C_t$$

The value function does not depend on time itself because the horizon is infinite and the problem is recursive

- We prefer to write the Bellman equation as:

$$V(K, Z) = \max_{C, K^+} U(C) + \beta E_{Z^+|Z} \left[V(K^+, Z^+)\right]$$

$$\text{s.t. } K^+ = Z\, F(K, 1) + (1-\delta)K - C \qquad \text{(S16a)}$$

DP: Choice problem in period t

- The first-order necessary condition for the maximization problem on the right-hand side is:

$$U'(C) = \beta E_{Z^+|Z}\left[\frac{\partial V(K^+, Z^+)}{\partial K^+}\right] \qquad \text{(S16b)}$$

- Using the Benveniste-Scheinkman Theorem results in:

$$\begin{aligned}\frac{\partial V(K, Z)}{\partial K} &= \beta E_{Z^+|Z}\left[\frac{\partial V(K^+, Z^+)}{\partial K^+} \cdot \Big(Z\, F_K(K, 1) + 1 - \delta\Big)\right] \\ &= \Big(Z\, F_K(K, 1) + 1 - \delta\Big) \cdot U'(C) \qquad \text{(S16c)}\end{aligned}$$

- By leading (S16c) by one period gives:

$$\frac{\partial V(K^+, Z^+)}{\partial K^+} = \Big(Z^+ F_K(K^+, 1) + 1 - \delta\Big) \cdot U'(C^+) \qquad \text{(S16d)}$$

DP: Choice problem in period t

- Finally, by combining (S16b) and (S16d) we obtain the stochastic Euler equation:

$$U'(C) = \beta E_{Z^+|Z}\left[\left(Z^+ F_K(K^+,1)+1-\delta\right)\cdot U'(C^+)\right] \quad \text{(S16e)}$$

- In Chapters 18 and 19 we show how models containing a stochastic Euler equation can be solved (approximately) to obtain the policy functions $\mathbf{C}(K,Z)$ and $\mathbf{K}^+(K,Z)$

Optimal risk sharing (1)

- construct a tractable **aggregate** model in a situation where individuals are faced with an inherently stochastic world
- Key concept: **complete markets** in dated contingent claims – so-called **Arrow-Debreu securities**
- Simple example of a multi-period endowment economy inhabited by a large number of individuals
- Dynamic endowment economy featuring a time horizon denoted by T
- There are I agents indexed by $i = 1, \ldots, I$
- In each period $t = 0, 1, \ldots, T$ there is a realization of a some stochastic event $s_t \in \mathcal{S}$
- Any trading among individuals occurs after s_0 is revealed, i.e. the initial state is a certainty

Optimal risk sharing (2)

- History of stochastic events up to and including period t by the vector $h^t \in \mathcal{H}^t$:

$$h^t \equiv (s_0, s_1, \ldots, s_t) \tag{S17a}$$

  - $h^t = (h^{t-1}, s_t)$
  - h^t is publicly and perfectly observable by all agents
  - unconditional probability of observing h^t by $\pi_t(h^t)$

$$\pi_t(h^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\ldots\pi(s_1|s_0)$$
$$\pi_t(h^t|h^\tau) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\ldots\pi(s_{\tau+1}|s_\tau)$$

  - $\mathcal{H}^t$, denoting the set of all possible histories at time t, typically becomes very large as time evolves

- Endowment of the non-storable commodity for agent i depends on s_t and is denoted by $y^i(s_t)$

Optimal risk sharing (3)

- Consumption of agent i at time t under history h^t is denoted by $c_t^i(h^t)$
- At time $t = 0$ each agent i chooses a life-time consumption plan denoted by $c^i = \{c_t^i(h^t)\}_{t=0}^{\infty}$
- Expected utility function of agent i is given by:

$$\Lambda(c^i) \equiv E_0 \left[\beta^t U(c_t^i)\right] = \sum_{t=0}^{T} \sum_{h^t \in \mathcal{H}^t} \beta^t U(c_t^i(h^t))\pi_t(h^t) \quad \text{(S17b)}$$

- Economy-wide resource constraint (for all $t = 0, 1, \ldots, T$ and $h^t \in \mathcal{H}^t$):

$$\sum_{i=1}^{I} c_t^i(h^t) = \sum_{i=1}^{I} y_t^i(h^t) \quad \text{(S17c)}$$

Optimal risk sharing (4)

- How does a social planner allocate risk in the endowment economy?
- **Negishi approach**: characterize the set of Pareto optimal allocations
- Objective function of the social planner:

$$\Omega_0 \equiv \sum_{i=1}^{I} \lambda_i \Lambda(c^i) \tag{S17d}$$

  - λ_i is the time-invariant Pareto weight that the planner assigns to agent i
  - Every agent matters to the planner, $\lambda_i > 0$
  - By normalization one can always ensure that $\sum_{i=1}^{I} \lambda_i = 1$
- Social planner chooses $c^i = \left\{ c_t^i(h^t) \right\}_{t=0}^{\infty}$ for all i in order to maximize Ω_0 subject to the resource constraints (S17c)

Optimal risk sharing (5)

- Lagrangian:

$$\mathcal{L}_0 \equiv \sum_{t=0}^{T} \sum_{h^t \in \mathcal{H}^t} \left[\sum_{i=1}^{I} \lambda_i \beta^t U(c_t^i(h^t))\pi_t(h^t) + \mu_t(h^t) \sum_{i=1}^{I} \left(y_t^i(h^t) - c_t^i(h^t) \right) \right]$$

where $\mu_t(h^t)$ is the Lagrange multiplier for the resource constraint at time t and history h^t

- The first-order necessary condition for $c_t^i(h^t)$:

$$\beta^t U'(c_t^i(h^t))\pi_t(h^t) = \frac{\mu_t(h^t)}{\lambda_i} \qquad \text{(S17e)}$$

- Note that (S17e) must hold for each t, h^t, and i

Optimal risk sharing (6)

- Comparing a benchmark individual – say agent 1 – with any other agent i we obtain from (S17e) that for given t and h^t:

$$\frac{U'(c_t^i(h^t))}{U'(c_t^1(h^t))} = \frac{\lambda_1}{\lambda_i} \tag{S17f}$$

This is the **Efficient Risk Sharing condition**

- From (S17f) we obtain:

$$c_t^i(h^t) = U'^{-1}(\lambda_1 U'(c_t^1(h^t))/\lambda_i) \tag{S17g}$$

- By substituting (S17g) into the resource constraint (S17c) we obtain:

$$\sum_{i=1}^{I} U'^{-1}(\lambda_1 U'(c_t^1(h^t))/\lambda_i) = \sum_{i=1}^{I} y_t^i(h^t) \tag{S17h}$$

Optimal risk sharing (7)

- By substituting (S17g) into the resource constraint (S17c) we obtain:

$$\sum_{i=1}^{I} U'^{-1}(\lambda_1 U'(c_t^1(h^t))/\lambda_i) = \sum_{i=1}^{I} y_t^i(h^t) \qquad \text{(S17h)}$$

  - $c_t^1(h^t)$ depends on the **aggregate** realized endowment at time t (right-hand side)
  - Example with two agents ($I = 2$) and a logarithmic felicity function ($U(x) = \ln x$):

$$c_t^1(h^t) = \lambda_1 \sum_{i=1}^{I} y_t^i(h^t), \quad c_t^2(h^t) = (1-\lambda_1) \sum_{i=1}^{I} y_t^i(h^t) \quad \text{(S17i)}$$

Optimal risk sharing: Decentalization (1)

- The Pareto optimal equilibrium can be decentralized provided the securities market is *complete*, i.e. individuals are able to trade a (potentially huge) set of claims on period t consumption contingent on history h^t with each other
  - At time $t = 0$ agents trade claims to consumption at all times $t > 0$ contingent on all possible histories h^t
  - Trading occurs at all nodes $h^t \in \mathcal{H}^t$ because the agents do not know which histories will actually materialize
  - After time $t = 0$ no further trades occur
  - We let $q_t^0(h^t)$ denote the price of claims on period t consumption contingent on history h^t
- Individual i's lifetime budget constraint:

$$\sum_{t=0}^{T} \sum_{h^t \in \mathcal{H}^t} q_t^0(h^t) c_t^i(h^t) = \sum_{t=0}^{T} \sum_{h^t \in \mathcal{H}^t} q_t^0(h^t) y_t^i(h^t) \qquad \text{(S18a)}$$

Optimal risk sharing: Decentalization (2)

- Individual i chooses $c^i \equiv \left\{c_t^i(h^t)\right\}_{t=0}^{\infty}$ in order to maximize (S17b) subject to (S18a)
- Lagrangian:

$$\mathcal{L}_0^i \equiv \sum_{t=0}^{T} \sum_{h^t \in \mathcal{H}^t} \left[\beta^t U(c_t^i(h^t))\pi_t(h^t) + \zeta_i q_t^0(h^t)\left(y_t^i(h^t) - c_t^i(h^t)\right)\right]$$

where ζ_i is the Lagrange multiplier for the lifetime budget constraint faced by agent i

- First-order necessary condition for $c_t^i(h^t)$:

$$\beta^t U'(c_t^i(h^t))\pi_t(h^t) = \zeta_i q_t^0(h^t) \tag{S18b}$$

- Compare agent 1 and any other agent i:

$$\frac{U'(c_t^i(h^t))}{U'(c_t^1(h^t))} = \frac{\zeta_i}{\zeta_1} \tag{S18c}$$

Optimal risk sharing: Decentalization (3)

- Hence in the decentralized economy we have:

$$c_t^i(h^t) = U'^{-1}(\zeta_i U'(c_t^1(h^t))/\zeta_1) \quad \text{(S18d)}$$

$$\sum_{i=1}^{I} U'^{-1}(\zeta_i U'(c_t^1(h^t))/\zeta_1) = \sum_{i=1}^{I} y_t^i(h^t) \quad \text{(S18e)}$$

- Just as in the planning optimum, $c_t^1(h^t)$ depends only on the aggregate realized endowment at time t (right-hand side)
- For the two-person logarithmic felicity case:

$$c_t^1(h^t) = \frac{\zeta_2}{\zeta_1 + \zeta_2} \sum_{i=1}^{I} y_t^i(h^t), \quad c_t^2(h^t) = \frac{\zeta_1}{\zeta_1 + \zeta_2} \sum_{i=1}^{I} y_t^i(h^t) \quad \text{(S18f)}$$

Optimal risk sharing: Decentalization (4)

- For the two-person logarithmic felicity case:

$$c_t^1(h^t) = \frac{\zeta_2}{\zeta_1 + \zeta_2} \sum_{i=1}^{I} y_t^i(h^t), \quad c_t^2(h^t) = \frac{\zeta_1}{\zeta_1 + \zeta_2} \sum_{i=1}^{I} y_t^i(h^t) \tag{S18f}$$

  - Intuitively, a "lucky individual" is somebody who at time $t = 0$ expects the economy to evolve in such a way that the value of the lifetime endowment is high (Mother Nature has stacked the deck in favour of such a person)
  - For such an individual the marginal utility of endowment income (ζ_i) is relatively low. Hence, if person 1 is the lucky individual then it follows that $\zeta_2 > \zeta_1$ and that $c_t^1(h^t) > c_t^2(h^t)$

Optimal risk sharing: Decentalization (5)

- Back to the Negishi (1960) insight
  - the competitive risk-sharing equilibrium allocation is Pareto optimal with weights such that $\lambda_i = 1/\zeta_i$
  - in view of the inverse relationship between λ_i and ζ_i, a "lucky individual" (as defined above) gets a larger weight in the social welfare function than a less lucky person gets
  - the shadow prices of the social planning problem equals the contingent price, i.e. $\mu_t(h^t) = q_t^0(h^t)$
- Some two-person examples
  - Only idiosyncratic risk: complete insurance
  - Only aggregate risk: efficient risk bearing

Optimal risk sharing: Decentalization (6)

- *Special Case #1*: only idiosyncratic risk
  - Stochastic events are such that $s_t \in [0, 1]$ and that the endowments are given by:

$$y_t^1(h^t) = s_t, \qquad y_t^2(h^t) = 1 - s_t \tag{S18g}$$

  - There is no aggregate risk because total endowment income is constant for each t and h^t, i.e. $\sum_{i=1}^{2} y_t^i(h^t) = 1$
  - Competitive risk-sharing equilibrium:

$$\bar{c}^i = (1-\beta) \sum_{t=0}^{T} \sum_{h^t \in \mathcal{H}^t} \beta^t \pi_t(h^t) y_t^i(h^t) \tag{S18h}$$

  - There is perfect consumption smoothing over time and across histories
  - Even though individual endowment incomes fluctuate randomly each individual can completely insure against this idiosyncratic risk

Optimal risk sharing: Decentalization (7)

- *Special Case #2*: only aggregate risk
  - Endowment incomes:

$$y_t^1(h^t) = \alpha s_t, \quad y_t^2(h^t) = (1-\alpha)s_t \tag{S18h}$$

    with $0 < \alpha < 1$
  - Agents are in the same boat
  - Competitive risk-sharing equilibrium:

$$c_t^2(h^t) = U'^{-1}(\zeta_2 U'(c_t^1(h^t))/\zeta_1) \tag{S18i}$$
$$c_t^1(h^t) + c_t^2(h^t) = s_t \tag{S18j}$$

  - Both consumption levels are stochastic
  - Efficient risk sharing results in shifting the risk to those who can best bear it

Aggregation (1)

- Under complete contingent-claim markets risk sharing is efficient and there is full insurance
- Under quite general conditions regarding preferences one can construct a fictional "representative agent" and ignore the underlying heterogeneity of individuals when interested in macroeconomic issues
- Simple demonstration of this result in our simple endowment model
- Set $q_0^0(h^0) = q_0^0(s_0) = 1$ as the numeraire (price system is expressed in units of period 0 goods)
- Logarithmic felicity function, $U(x) = \ln x$
- It follows from (S18b) that:

$$\zeta_i = \frac{1}{c_0^i(s_0)} = \frac{\beta^t \pi_t(h^t)}{q_t^0(h^t) c_t^i(h^t)} \tag{S19a}$$

Aggregation (2)

- Individual Euler equation:

$$\frac{c_t^i(h^t)}{c_0^i(s_0)} = \frac{\beta^t \pi_t(h^t)}{q_t^0(h^t)} \tag{S19b}$$

- The right-hand side of this expression is the same for all $i = 1, 2, \ldots, I$
- Aggregate Euler equation:

$$\frac{C_t(h^t)}{C_0(s_0)} = \frac{\beta^t \pi_t(h^t)}{q_t^0(h^t)} \tag{S19c}$$

where aggregate consumption, $C_t(h^t)$, is defined as:

$$C_t(h^t) \equiv \sum_{i=1}^{I} c_t^i(h^t) \tag{S19d}$$

Aggregation (3)

- Consider the "representative agent" who has a utility function which depends on aggregate consumption:

$$\Lambda(C) \equiv E_0 \left[\beta^t U(C_t)\right] = \sum_{t=0}^{T} \sum_{h^t \in \mathcal{H}^t} \beta^t \ln C_t(h^t) \pi_t(h^t) \quad \text{(S19e)}$$

- The economy-wide budget constraint facing the representative agent is obtained by summing (S18a) over all individuals:

$$\sum_{t=0}^{T} \sum_{h^t \in \mathcal{H}^t} q_t^0(h^t) C_t(h^t) = \sum_{t=0}^{T} \sum_{h^t \in \mathcal{H}^t} q_t^0(h^t) Y_t(h^t) \quad \text{(S19f)}$$

where $Y_t(h^t)$ is the aggregate endowment:

$$Y_t(h^t) \equiv \sum_{i=1}^{I} y_t^i(h^t) \quad \text{(S19g)}$$

Aggregation (4)

- The fictional representative agent chooses $C_t(h^t)$ in order to maximize (S19e) subject to (S19f)
- Lagrangian:

$$\mathcal{L} \equiv \sum_{t=0}^{T} \sum_{h^t \in \mathcal{H}^t} \left[\beta^t \ln C_t(h^t) \pi_t(h^t) + \zeta q_t^0(h^t) \Big(Y_t(h^t) - C_t(h^t) \Big) \right],$$

where ζ is the Lagrange multiplier for the aggregate budget constraint

- First-order necessary condition for $C_t(h^t)$:

$$\beta^t \frac{\pi_t(h^t)}{C_t(h^t)} = \zeta q_t^0(h^t) \tag{S19h}$$

Aggregation (5)

- First-order necessary condition for $C_t(h^t)$:

$$\beta^t \frac{\pi_t(h^t)}{C_t(h^t)} = \zeta q_t^0(h^t) \tag{S19h}$$

  - It is easy to see that (S19h) implies (S19c)
  - We obtain exactly the same solution for $C_t(h^t)$ as before by letting the fictional agent do the utility maximization
  - The economy's aggregate consumption $C_t(h^t)$ behaves *as if* chosen by a representative consumer with a logarithmic felicity function defined over aggregate consumption, $C_t(h^t)$, who owns the economy's total endowment $Y_t(h^t)$
- Aggregation result holds for all felicity functions of the HARA class