Foundations of Modern Macroeconomics
Third Edition

Chapter 14: Endogenous economic growth

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Outline

Aims of this chapter (1)

- Study the main theories of endogenous growth
- *Key notion*: Can we devise a growth theory in which the steady-state growth rate is *endogenous*, i.e. depends not just on exogenous things like the population growth rate and the rate of Harrod-neutral technological change?
- Can we open up the black box of technological change?
- Following the influential work of Paul Romer in the mid 1980s a very active research field has developed

Aims of this chapter (2)

- We will give a selective overview of this huge body of literature Three groups can be distinguished:
  - "Capital fundamentalist" models. Physical capital forms the engine of growth
  - Human capital formation. The knowledge inside human heads is crucial to growth. People accumulate knowledge with good purpose
  - Endogenous technology. Profit-seeking firms engage in research & development to make new products or services, or devise new production processes

Recall the Inada conditions

- In exogenous growth models there exist diminishing returns to capital
- As $k(t)$ rises over time, the average product of capital falls:

$$\frac{d\left[f(k(t))/k(t)\right]}{dk(t)} = -\frac{\left[f(k(t)) - k(t)f'(k(t))\right]}{k(t)^2} < 0 \qquad \text{(S1)}$$

- Property (S1) necessary but not sufficient for existence of steady-state capital-labour ratio
- Inada conditions are strong enough:

$$\lim_{k(t)\to 0} \frac{f(k(t))}{k(t)} = \lim_{k(t)\to 0} \frac{f'(k(t))}{1} = \infty \qquad \text{(S2)}$$

$$\lim_{k(t)\to \infty} \frac{f(k(t))}{k(t)} = \lim_{k(t)\to \infty} \frac{f'(k(t))}{1} = 0 \qquad \text{(S3)}$$

Inada conditions violated

- *Key idea*: There are perfectly reasonable production functions which do not satisfy the Inada conditions
- Take, for example, the CES production function:

$$F(K(t), L(t)) \equiv Z \cdot \left[\alpha K(t)^{1/\xi} + (1-\alpha) L(t)^{1/\xi}\right]^{\xi} \quad \Leftrightarrow$$

$$f(k(t)) \equiv Z \cdot \left[1 - \alpha + \alpha k(t)^{1/\xi}\right]^{\xi}$$

where ξ is a coefficient involving the substitution elasticity between capital and labour, σ_{KL}:

$$\xi \equiv \frac{\sigma_{KL}}{\sigma_{KL} - 1}$$

- The average product of capital (APK) equals:

$$\frac{f(k(t))}{k(t)} = Z \cdot \left[(1-\alpha) k(t)^{-1/\xi} + \alpha\right]^{\xi} \qquad \text{(S4)}$$

CES production function (1)

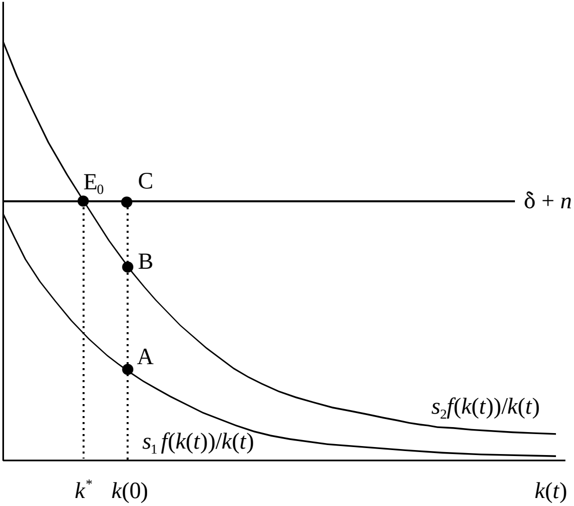
- Two cases must be distinguished:
  - **Case A**: Difficult substitution between K and L ($0 < \sigma_{KL} < 1$ and $\xi < 0$)
  - **Case B**: Easy substitution between K and L ($\sigma_{KL} > 1$ and $\xi > 1$)
- **Case A**. With difficult substitution we obtain from (S4):

$$0 < \lim_{k(t)\to 0} \frac{f(k(t))}{k(t)} = Z \cdot \alpha^{\xi} < \infty$$

$$\lim_{k(t)\to\infty} \frac{f(k(t))}{k(t)} = Z \cdot \lim_{k(t)\to\infty} \frac{f'(k(t))}{1} = 0$$

The APK is finite for $k(t) \to 0$. Hence, it may not be even high enough to sustain a non-trivial steady state (see **Figure 14.1**)

Figure 14.1: Difficult substitution between K and L



CES production function (2)

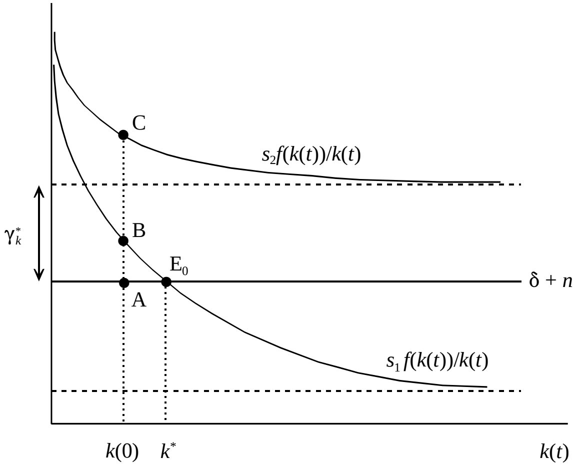
- **Case B**. With easy substitution we derive from (S4):

$$\lim_{k(t)\to 0} \frac{f(k(t))}{k(t)} = Z \cdot \lim_{k(t)\to 0} \frac{f'(k(t))}{1} = \infty$$

$$\lim_{k(t)\to \infty} \frac{f(k(t))}{k(t)} = Z \cdot \alpha^{\xi} > 0$$

There is a lower bound on the APK as $k(t) \to \infty$ Hence, there may be perpetual growth in $k(t)$ (see **Figure 14.2**)

Figure 14.2: Easy substitution between K and L



CES production function (3)

- The asymptotic growth rate is:

$$\gamma^* = sZ\alpha^{\sigma_{KL}/(\sigma_{KL}-1)} - (\delta + n) > 0$$

- We call γ^* an *endogenous* growth rate because the savings rate, s, affects it!
- Even though there are diminishing returns to capital, K and L substitute easily. Hence, labour does not become an effective constraint. Scarce labour is substituted by capital indefinitely
- But, the share of labour goes to zero – contra SF3 and SF5

Macroeconomic technology linear in capital

- An even more radical model is the so-called *AK model*
- macroeconomic technology is:

$$Y(t) = Z \cdot K(t) \tag{S5}$$

- The MPK is constant and labour is eliminated from the model altogether!
- Eq. (S5) is not as silly as one might think
- Two ways to rationalize this macroeconomic relationship
  - There exist external productivity effects between individual firms
  - Public infrastructure causes external productivity effects on individual firms

Inter-firm technological externalities (1)

- *Key idea*: individual firms experience diminishing returns to labour and capital. But external effects between firms render the marginal product of aggregate capital constant
- Technology available to firm i:

$$Y_i\,(t) = F(K_i(t), L_i(t)) \equiv Z\,(t)\,K_i\,(t)^{\alpha}\,L_i\,(t)^{1-\alpha} \qquad \text{(S6)}$$

with $0 < \alpha < 1$. Here Y_i, K_i, and L_i, stand for, respectively, output, capital input, and labour input of firm i $(= 1, \cdots, N_0)$, and N_0 is the fixed number of firms. $Z\,(t)$ is the general level of factor productivity which is taken as given by the individual firm

Inter-firm technological externalities (2)

- Firm i's objective function:

$$V_i(0) = \int_0^{\infty} \Big[F(K_i(t), L_i(t)) - w(t) L_i(t) - (1 - s_I)\, I_i(t) \Big] e^{-R(t)} dt$$

where $R(t) \equiv \int_0^t r(\tau) d\tau$ is the cumulative discount factor, and s_I is the investment subsidy

- Marginal productivity conditions for labour and capital:

$$w\,(t) = F_L(K_i(t), L_i(t)) = (1-\alpha)\, Z\,(t)\, k_i\,(t)^{\alpha} \qquad \text{(S7)}$$

$$R^K(t) = F_K(K_i(t), L_i(t)) = \alpha Z\,(t)\, k_i\,(t)^{\alpha-1} \qquad \text{(S8)}$$

- Rental rate of capital:

$$R^K\,(t) \equiv (r\,(t) + \delta)\,(1 - s_I)$$

Inter-firm technological externalities (3)

- Symmetric solution: the rental rate on each factor is the same for all firms, i.e. they all choose the same capital intensity and $k_i(t) = k(t)$ for all $i = 1, \cdots, N_0$. Aggregation over firms simple
- Inter-firm externality takes the following form:

$$Z(t) = z_0 K(t)^{1-\alpha}, \qquad z_0 > 0 \tag{S9}$$

where $K(t) \equiv \sum_i K_i(t)$ is the aggregate capital stock
- With a fixed macro labour supply, using (S9) in (S6)–(S8) results in:

$$Y(t) = Z_0 K(t) \tag{S10}$$

$$w(t) L_0 = (1-\alpha) Y(t) \tag{S11}$$

$$R^K(t) = \alpha Z_0 \tag{S12}$$

where $Y(t) \equiv \sum_i Y_i(t)$ is aggregate output $(Z_0 \equiv z_0 L_0^{1-\alpha})$

Inter-firm technological externalities (4)

- The national income share of labour is positive and there are constant returns to capital at the macroeconomic level. This result follows from the fact that the exponents for K_i in (S6) and for K in (S9) **precisely** add up to unity
- Household side: Ramsey model with fixed labour supply L_0
- Infinitely-lived representative household

$$\Lambda(0) = \int_0^\infty \frac{C(t)^{1-1/\sigma} - 1}{1 - 1/\sigma} e^{-\rho t} dt$$

$$\dot{A}(t) = r(t)A(t) + w(t)L_0 - (1 + t_C)C(t) - T(t)$$

- Euler equation:

$$\frac{\dot{C}(t)}{C(t)} = \sigma \cdot (r(t) - \rho)$$

Inter-firm technological externalities (5)

- Closed economy
- No government debt. Government consumption $G(t) = gY(t)$
  Government budget equation:

$$T(t) + t_C C(t) = G(t) + s_I I(t)$$

- The only financial asset which can be accumulated consists of company shares. Replacement value of capital equals $1 - s_I$, so $A(t) = [1 - s_I(t)] K(t)$
- The key equations of the basic *AK* growth model have been summarized in **Table 14.1**

Table 14.1: An AK growth model with inter-firm external effects

$$\frac{\dot{C}(t)}{C(t)} = \sigma \cdot (r(t) - \rho) \quad \text{(T1.1)}$$

$$\dot{K}(t) = [(1-g) \cdot Z_0 - \delta] \cdot K(t) - C(t) \quad \text{(T1.2)}$$

$$r(t) = \frac{\alpha Z_0}{1 - s_I} - \delta \quad \text{(T1.3)}$$

Main properties of the model with inter-firm technological externalities

- The growth rate in the economy is:

$$\gamma^* = \frac{\dot{C}(t)}{C(t)} = \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{K}(t)}{K(t)} = \frac{\dot{I}(t)}{I(t)} = \sigma \left[\frac{\alpha Z_0}{1 - s_I} - \delta - \rho \right]$$

- Endogenous growth: the policy maker can affect it by setting s_I. Indeed, $d\gamma^*/ds_I > 0$. In **Figure 14.3**, $\theta \equiv C/K$ falls. The lump-sum tax increase makes people poorer
- Consumption tax t_C and government consumption g do not affect growth rate
- Growth path is not Pareto-efficient. Firms fail to take inter-firm externality into account
- No transitional dynamics (for case with $\dot{s}_I(t) = 0$)

Why is there no transitional dynamics in this model? (1)

- Define $\theta(t) \equiv C(t)/K(t)$ and note that:

$$\frac{\dot{\theta}(t)}{\theta(t)} = \frac{\dot{C}(t)}{C(t)} - \frac{\dot{K}(t)}{K(t)}$$

- Use (T1.1)-(T1.3) to find:

$$\frac{\dot{\theta}(t)}{\theta(t)} = \theta(t) - \theta^* \qquad \text{(A)}$$

where θ^* is defined as:

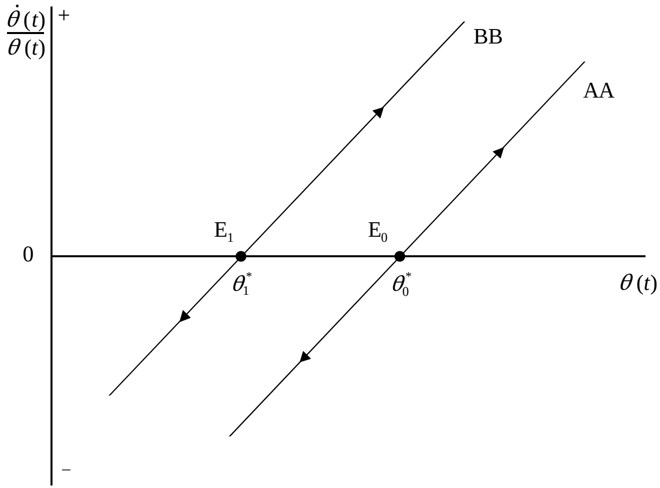
$$\theta^* \equiv (1-g)\,Z_0 - \delta + \sigma(\rho+\delta) - \frac{\alpha\sigma Z_0}{1-s_I} > 0.$$

- Equation (A) is an unstable differential equation for which the only economically feasible solution is the steady-state, i.e. $\theta(t) = \theta^*$. See **Figure 14.3**

Why is there no transitional dynamics in this model? (2)

- Hence the capital stock, investment, and output, must feature the same growth rate as consumption
- The level of the different variables can be determined by using the initial condition regarding the capital stock and noting that $C(0) = \theta^* K(0)$
- In the absence of shocks in the interval $(0, t)$, we thus find that $K(t) = K(0) e^{\gamma^* t}$, $C(t) = \theta^* K(t)$, $Y(t) = Z_0 K(t)$, etcetera
- The *growth rate* of the economy can be permanently affected by the investment subsidy! In **Figure 14.3** an increase in s_I shifts the equilibrium from θ_0^* to θ_1^*. The lump-sum tax increase needed to finance the higher subsidy makes people poorer

Figure 14.3: Consumption-capital ratio



Public infrastructural capital (1)

- Barro suggest a model in which productive government spending affects productivity (and growth). Technology is still as in (S6) but there is an output tax, t_Y
- The productivity conditions for individual firms are:

$$w(t) = (1-\alpha)(1-t_Y) Z(t) k_i(t)^{\alpha}$$
$$r(t) + \delta_k = \alpha(1-t_Y) Z(t) k_i(t)^{\alpha-1}$$

- In the *spirit* of Barro's model we assume:

$$Z(t) = z_0 K_G(t)^{1-\alpha} \quad \text{(S13)}$$

where $K_G(t)$ is the *stock* of public capital, consisting of infrastructural objects like roads, airports, bridges, and the like

Public infrastructural capital (2)

- Aggregating over all firms i gives:

$$Y(t) = Z_0 K(t)^{\alpha} K_G(t)^{1-\alpha} \quad \text{(S14)}$$

$$w(t) L_0 = (1-\alpha)(1-t_Y) Y(t) \quad \text{(S15)}$$

$$r(t) + \delta_k = \alpha (1-t_Y) Z_0 \left(\frac{K_G(t)}{K(t)} \right)^{1-\alpha} \quad \text{(S16)}$$

  - Diminishing returns to private capital also at the macro level, but...
  - If the government maintains constant K_G/K ratio then model is like *AK* model

Public infrastructural capital (3)

- Accumulation equation for public capital:

$$\dot{K}_G(t) = I_G(t) - \delta_g K_G(t)$$

where $I_G(t)$ is the **flow** of public investment (exogenous), and δ_g is the depreciation rate of public capital

- The government budget constraint is:

$$t_Y Y(t) = I_G(t) + gY(t)$$

- The key equations of the model have been summarized in **Table 14.2**

Table 14.2: An AK growth model with public capital

$$\frac{\dot{C}(t)}{C(t)} = \sigma \cdot [r(t) - \rho] \tag{T2.1}$$

$$\frac{\dot{K}(t)}{K(t)} = (1 - t_Y) Z_0 \left(\frac{K(t)}{K_G(t)} \right)^{\alpha - 1} - \frac{C(t)}{K(t)} - \delta_k \tag{T2.2}$$

$$\frac{\dot{K}_G(t)}{K_G(t)} = (t_Y - g) Z_0 \left(\frac{K(t)}{K_G(t)} \right)^{\alpha} - \delta_g \tag{T2.3}$$

$$r(t) = \alpha (1 - t_Y) Z_0 \left(\frac{K(t)}{K_G(t)} \right)^{\alpha - 1} - \delta_k \tag{T2.4}$$

Public infrastructural capital (4)

- The steady-state growth rate is γ^*:

$$\gamma^* = \sigma\left[r^* - \rho\right] \quad \text{(A)}$$

$$\gamma^* = (1 - t_Y)\, Z_0\, (\kappa^*)^{\alpha-1} - \theta^* - \delta_k \quad \text{(B)}$$

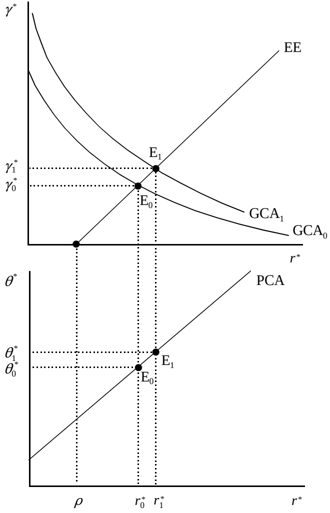
$$\gamma^* = (t_Y - g)\, Z_0\, (\kappa^*)^{\alpha} - \delta_g \quad \text{(C)}$$

$$r^* = \alpha\,(1 - t_Y)\, Z_0\, (\kappa^*)^{\alpha-1} - \delta_k \quad \text{(D)}$$

- In **Figure 14.4** we illustrate the nature of the solution *for a given tax rate* t_Y
  - EE is the Euler Equation (A)
  - GCA is the Government Capital Accumulation locus (solve (D) for κ^* and substitute in (C)
  - PCA is the Private Capital Accumulation locus (solve (D) for κ^* and substitute in (B)

Figure 14.4: Steady-state growth

Effect of a decrease in g



Public infrastructural capital (5)

- Implicit expression for γ^*:

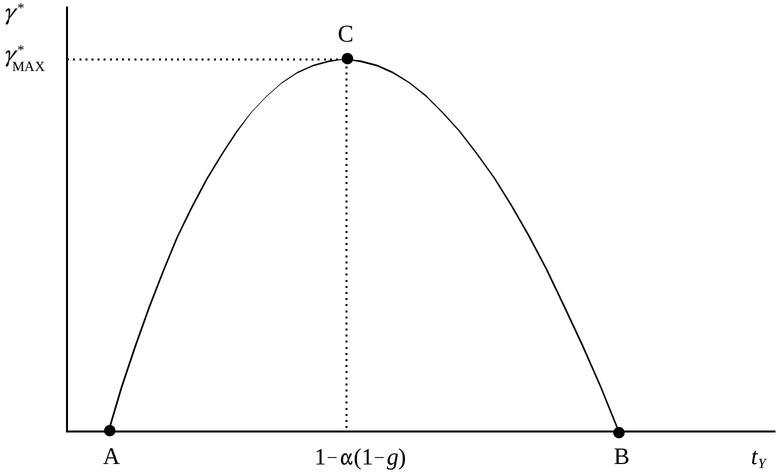
$$\gamma^* + \delta_g = \left(\alpha^\alpha Z_0\right)^{1/(1-\alpha)} \left(t_Y - g\right) \left(\frac{\sigma\left(1 - t_Y\right)}{\gamma^* + \sigma\left(\rho + \delta_k\right)}\right)^{\alpha/(1-\alpha)}$$

- Slope of the growth line:

$$\frac{t_Y - g}{\gamma^* + \delta_g} \cdot \frac{d\gamma^*}{dt_Y} = \frac{1 - \frac{\alpha}{1-\alpha}\frac{t_Y-g}{1-t_Y}}{1 + \frac{\alpha}{1-\alpha}\frac{\gamma^*+\delta_g}{\gamma^*+\sigma(\rho+\delta_k)}}$$

- In **Figure 14.5** we plot the steady-state growth rate as a function of the output tax, t_Y
- This *AK* model features non-trivial transitional dynamics

Figure 14.5: Productive government spending and growth



Transitional dynamics in the infrastructural model (1)

- Consider the model in **Table 14.2**
- Define: $\theta(t) \equiv C(t)/K(t)$ and $\kappa(t) \equiv K(t)/K_G(t)$
- Rewrite the system:

$$\frac{d\ln\theta(t)}{dt} = \sigma\left[r(t)-\rho\right] - (1-t_Y)\,Z_0\kappa(t)^{\alpha-1} + \theta(t) + \delta_k$$

$$\frac{d\ln\kappa(t)}{dt} = (1-t_Y)\,Z_0\kappa(t)^{\alpha-1} - (t_Y-g)\,Z_0\kappa(t)^{\alpha} - \theta(t) + \delta_g - \delta_k$$

$$r(t) = \alpha(1-t_Y)\,Z_0\kappa(t)^{\alpha-1} - \delta_k$$

Transitional dynamics in the infrastructural model (2)

- In order to study the dynamic properties of the model, we loglinearize it around the steady-state point (θ^*, κ^*) to obtain:

$$\begin{bmatrix} \frac{d\ln\theta(t)}{dt} \\ \frac{d\ln\kappa(t)}{dt} \end{bmatrix} = \Delta \cdot \begin{bmatrix} \ln\theta(t) - \ln\theta^* \\ \ln\kappa(t) - \ln\kappa^* \end{bmatrix}$$

- Δ is the Jacobian matrix:

$$\Delta \equiv \begin{bmatrix} \theta^* & \frac{(1-\alpha)(1-\alpha\sigma)(r^*+\delta_k)}{\alpha} \\ -\theta^* & -\frac{(1-\alpha)(r^*+\delta_k)+\alpha^2(\gamma^*+\delta_g)}{\alpha} \end{bmatrix}$$

Transitional dynamics in the infrastructural model (3)

- The determinant of Δ is given by:

$$|\Delta| \equiv -\theta^* \Big[(1-\alpha) \, \sigma \, (r^* + \delta_k) + \alpha \, (\gamma^* + \delta_g) \Big] < 0$$

- There is one negative (stable root) $-\lambda_1 < 0$ and one positive (unstable) root, $\lambda_2 > 0$, and the model is saddle-path stable
- $\theta(t)$ is a jumping variable (because $C(t)$ is) whilst $\kappa(t)$ is predetermined (because $K(t)$ and $K_G(t)$ are)
- Given initial values $K(0)$ and $K_G(0)$ (and thus for $\kappa(0) \equiv K(0) / K_G(0)$), the model converges along the saddle path toward the steady-state equilibrium
- The transition speed is equal to the absolute value of the stable root, λ_1

Human capital accumulation as the engine of growth

- *Key idea*: (Uzawa) all technical knowledge is embodied in labour. Educational sector uses labour to augment the state of knowledge in the economy
- Uzawa (1965) assumed:

$$\frac{\dot{Z}_L}{Z_L} = \Psi\left(\frac{L_E}{L}\right)$$

  where Z_L is labour-augmenting technical progress and L_E is labour used in the educational sector $(\Psi' > 0 > \Psi'')$
- Basic idea was taken over by Lucas (1988). He interprets Z_L as human capital ("skills") and calls it H. Rational agents accumulate human capital by dedicating some of their time on education (hence, the name of the model: "learning or doing")

Lucas-Uzawa model (1)

- Human capital accumulation function:

$$\frac{\dot{H}(t)}{H(t)} = Z_E \frac{L_E(t)}{L(t)} - \delta_h \tag{S14}$$

where Z_E is a positive constant

- Lifetime utility of the representative household:

$$\Lambda(0) = \int_0^\infty \frac{C(t)^{1-1/\sigma} - 1}{1 - 1/\sigma} e^{-\rho t} dt \tag{S15}$$

- Time constraint of the household:

$$L_E(t) + L_P(t) = L_0 \tag{S16}$$

where L_P is time spent working ("doing") rather than going to school

Lucas-Uzawa model (2)

- Aggregate production function:

$$Y(t) = F(K(t), N_P(t)) = Z_Y N_P(t)^{1-\alpha} K(t)^{\alpha}$$
$$N_P(t) \equiv H(t) L_P(t)$$

- Factors receive their respective marginal products:

$$\begin{aligned} R^K(t) &= F_K(K(t), N_P(t)) = \alpha Z_Y k(t)^{\alpha-1} \\ w(t) &= H(t) \cdot F_N(K(t), N_P(t)) \\ &= (1-\alpha) Z_Y H(t) \cdot k(t)^{\alpha} \end{aligned} \tag{S17}$$

with $k(t) \equiv K(t)/N_P(t)$

- Equation (S17) shows that the wage rises with the skill level. Household has an incentive to accumulate human capital

Lucas-Uzawa model (3)

- The household takes $k(t)$ and thus F_N and F_K as given. These are **macro** variables
- The household chooses sequences for consumption and the stocks of physical and human capital in order to maximize lifetime utility (S15) subject to:
  - the time constraint (S16)
  - the accumulation identity for physical capital, $\dot{K}(t) = I(t) - \delta_k K(t)$
  - the budget identity:

$$I(t) + C(t) + T(t) = w(t)L_P(t) + R^K(t)K(t) + s_E w(t) L_E(t)$$

where $T(t)$ is a lump-sum tax and s_E is a time-invariant education subsidy received from the government ($\dot{s}_E = 0$)

Lucas-Uzawa model (4)

- Current-value Hamiltonian:

$$\mathcal{H}_C(t) = \frac{C(t)^{1-1/\sigma} - 1}{1 - 1/\sigma} + \mu_H(t) \cdot \left[Z_E \frac{L_E(t)}{L_0} - \delta_h \right] \cdot H(t)$$
$$+ \mu_K(t) \cdot \left[\left(R^K(t) - \delta_k \right) K(t) + H(t) F_N(k(t), 1)(L_0 - L_E(t)) \right.$$
$$\left. + s_E H(t) F_N(k(t), 1) L_E(t) - C(t) - T(t) \right]$$

where $\mu_K(t)$ and $\mu_H(t)$ are the co-state variables

- First-order necessary conditions:

$$C(t)^{-1/\sigma} = \mu_K(t)$$
$$\mu_H(t) \frac{Z_E}{L_0} = \mu_K(t)(1 - s_E) F_N(k(t), 1)$$
$$\frac{\dot{\mu}_K(t)}{\mu_K(t)} = \rho + \delta_k - F_K(k(t), 1)$$
$$\frac{\dot{\mu}_H(t)}{\mu_H(t)} = \rho + \delta_h - Z_E \frac{L_E(t)}{L_0} - \frac{\mu_K(t)}{\mu_H(t)} [L_0 - (1 - s_E) L_E(t)] F_N(k(t), 1)$$
$$0 = \lim_{t\to\infty} e^{-\rho t} \mu_K(t) K(t) = \lim_{t\to\infty} e^{-\rho t} \mu_H(t) H(t)$$

Lucas-Uzawa model (5)

- Fundamental principle of valuation: the rate of return on different assets with the same riskiness must be equalized
- For each asset the rate of return can be computed as the sum of dividends plus capital gains divided by the price of the asset:

$$\rho = \frac{\dot{\mu}_K(t) + D_K(t)}{\mu_K(t)} = \frac{\dot{\mu}_H(t) + D_H(t)}{\mu_H(t)}$$

where $D_K(t)$ and $D_H(t)$ are "dividend payments" on physical and human capital, respectively:

$$D_K(t) \equiv \mu_K(t) \left[F_K(k(t), 1) - \delta_k \right]$$

$$D_H(t) \equiv \mu_H(t) \left[\frac{Z_E}{1 - s_E} - \delta_h \right]$$

- The key expressions of the Lucas-Uzawa model are gathered in **Table 14.3** ($p(t) \equiv \mu_H(t) / \mu_K(t)$ is the relative shadow price of human capital)

Table 14.3: The Lucas-Uzawa model of growth and human capital accumulation

$$\frac{\dot{p}(t)}{p(t)} = r(t) + \delta_h - \frac{Z_E}{1-s_E} \tag{T3.1}$$

$$\frac{\dot{C}(t)}{C(t)} = \sigma\left[r(t) - \rho\right] \tag{T3.2}$$

$$\frac{\dot{K}(t)}{K(t)} = (1-g)\, Z_Y k(t)^{\alpha-1} - \frac{C(t)}{K(t)} - \delta_k \tag{T3.3}$$

$$\frac{\dot{H}(t)}{H(t)} = Z_E l_E(t) - \delta_h \tag{T3.4}$$

$$p(t) = (1-s_E)(1-\alpha)\frac{Z_Y L_0}{Z_E} k(t)^{\alpha} \tag{T3.5}$$

$$k(t) \equiv \frac{K(t)}{\left[1 - l_E(t)\right] L_0 H(t)} \tag{T3.6}$$

$$r(t) \equiv \alpha Z_Y k(t)^{\alpha-1} - \delta_k \tag{T3.7}$$

Steady-state growth in the Lucas-Uzawa model (1)

- Balanced growth path easy to analyze
- Define $\theta(t) \equiv C(t)/K(t)$ and $\kappa(t) \equiv K(t)/H(t)$
- Along the balanced growth path, consumption and the stocks of physical and human capital all grow at the same exponential growth rate, γ^*, so that $\theta(t) = \theta^*$ and $\kappa(t) = \kappa^*$. Also $p(t) = p^*$, $l_E(t) = l_E^*$, $k(t) = k^*$, and $r(t) = r^*$. The steady state can be solved recursively
- **Step 1**: Equation (T3.1) fixes the steady-state interest rate:

$$r^* = \frac{Z_E}{1 - s_E} - \delta_h$$

Steady-state growth in the Lucas-Uzawa model (2)

- **Step 2**: Given r^*, (T3.2) and (T3.7) determine, respectively, γ^* and k^*:

$$\gamma^* = \sigma \cdot [r^* - \rho] = \sigma \cdot \left[\frac{Z_E}{1 - s_E} - \delta_h - \rho\right] \qquad \text{(S18)}$$

$$k^* = \left(\frac{\alpha Z_Y}{r^* + \delta_k}\right)^{1/(1-\alpha)}$$

- **Step 3**: Given γ^* and k^* we find from (T3.3)-(T3.5):

$$\theta^* = \frac{1-g}{\alpha}(r^* + \delta_k) - \gamma^* - \delta_k$$

$$l_E^* = \frac{\gamma^* + \delta_h}{Z_E}$$

$$p^* = (1 - s_E)(1 - \alpha)\frac{Z_Y L_0}{Z_E}(k^*)^\alpha$$

Steady-state growth in the Lucas-Uzawa model (3)

- **Step 4**: Given k^* and l_E^* we obtain from (T3.6):

$$\kappa^* = k^* \cdot [1 - l_E^*] \cdot L_0$$

- **Step 5**: It remains to be checked that the (common) growth rate given in (S18) is actually feasible ($l_E^* < 1$). The feasibility requirement thus places an upper limit on the allowable intertemporal substitution elasticity:

$$\sigma < \frac{Z_E - \delta_h}{Z_E / (1 - s_E) - (\rho + \delta_h)}$$

Transitional dynamics in the Lucas-Uzawa model (1)

- In esssence only three fundamental dynamic variables in Table 14.3: $p(t)$, $\theta(t) \equiv C(t)/K(t)$, and $\kappa(t) \equiv K(t)/H(t)$
- Two quasi-reduced-form relationships
- **Relationship 1**. It follows from (T3.5) that $k(t)$ is an increasing function of both $p(t)$ and s_E:

$$k(t) = \left(\frac{Z_E p(t)}{(1-\alpha) Z_Y L_0 (1-s_E)} \right)^{1/\alpha} \equiv \Psi(\underset{+}{p(t)}, \underset{+}{s_E}) \quad \text{(S19)}$$

- **Relationship 2**. We find from (T3.6) that $l_E(t)$ depends negatively on $\kappa(t)$ and positively on $k(t)$ (and thus, via (S19), on $p(t)$ and s_E):

$$l_E(t) = 1 - \frac{\kappa(t)}{L_0 \Psi(p(t), s_E)} \quad \text{(S20)}$$

Transitional dynamics in the Lucas-Uzawa model (2)

- Hence, it follows from (S19)–(S20) that $k(t)$ and $l_E(t)$ are uniquely determined by the fundamental state variables, $p(t)$ and $\kappa(t)$
- In order to study the dynamic properties of the model, we log-linearize it around the steady-state point (θ^*, κ^*) to obtain:

$$\begin{bmatrix} \frac{d \ln p(t)}{dt} \\ \frac{d \ln \theta(t)}{dt} \\ \frac{d \ln \kappa(t)}{dt} \end{bmatrix} = \Delta \cdot \begin{bmatrix} \ln p(t) - \ln p^* \\ \ln \theta(t) - \ln \theta^* \\ \ln \kappa(t) - \ln \kappa^* \end{bmatrix}$$

Transitional dynamics in the Lucas-Uzawa model (3)

- Δ is the Jacobian matrix:

$$\Delta \equiv \begin{bmatrix} -\frac{(1-\alpha)(r^*+\delta_k)}{\alpha} & 0 & 0 \\ -\frac{(1-\alpha)\sigma(r^*+\delta_k)+Z_E\left(1-l_E^*\right)}{\alpha} & 0 & Z_E\left(1-l_E^*\right) \\ -\frac{(1-\alpha)(1-g)(r^*+\delta_k)+\alpha Z_E\left(1-l_E^*\right)}{\alpha^2} & -\theta^* & Z_E\left(1-l_E^*\right) \end{bmatrix}$$

Transitional dynamics in the Lucas-Uzawa model (4)

- The determinant of Δ is given by:

$$|\Delta| \equiv -\frac{(1-\alpha)\left(r^{*}+\delta_{k}\right) Z_{E}\left(1-l_{E}^{*}\right) \theta^{*}}{\alpha}<0$$

so it follows that the product of the characteristic roots of Δ is negative, i.e. there is an odd number of negative roots

- In the text we prove saddle-point stability: one stable root $(-\lambda_1 < 0)$ and two unstable roots $(\lambda_2 > 0$ and $\lambda_3 > 0)$
- The model features two jumping variables ($p(t)$ and $\theta(t)$) and one predetermined (sticky) variable ($\kappa(t)$). The adjustment speed in the economy is given by λ_1. Given initial values for $K(0)$ and $H(0)$ (and thus for $\kappa(0) \equiv K(0)/H(0)$), the model converges along the saddle path toward the steady-state equilibrium

R & D as the engine of growth

- *Key idea*: Purposeful conduct of R&D activities is the source of growth
- In the absence of human and physical capital, households can nevertheless save by accumulating *patents*
- Patents are blueprints for the production of "slightly unique" products
- The patent holder has a little bit of monopoly power which can be exploited
- Hence, in this literature we leave the competitive framework and enter the realm of monopolistic competition. (Schumpeterian models of "creative destruction" can be built along the lines of the present model. Example: RIQ model.)

Overview of the expanding input variety (EIV) model

- Three productive sectors
- *Final goods sector* (CRTS, perfectly competitive, external effect "returns to specialization"): Produces a homogenous good using differentiated inputs in the production process
- *Intermediate goods sector* (many small monopolistically competitive firms): Each firm (patent holder) uses labour to produce its own slightly unique variety of the intermediate input
- *R&D sector* (CRTS, perfectly competitive, external effect "standing on the shoulder of giants"): Produces blueprints for new intermediate inputs, using labour as an input
- Production factors perfectly mobile

Final goods sector (1)

- *Final goods sector* (CRTS, perfectly competitive, external effect "returns to specialization"): Produces a homogenous good using differentiated inputs in the production process
- Technology:

$$Y(t) \equiv N(t)^{\eta} \cdot \left[\frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i(t)^{1/\mu} \right]^{\mu} \qquad \text{(S1)}$$

  where X_i is intermediate input i, N is the existing number of varieties, and μ and η are parameters ($\mu > 1$ and $\eta \geq 1$)
- If $\eta > 1$ there are returns to specialization as in Adam Smith's pin factory. If $N \uparrow$ then inputs are more finely differentiated and firms can use a more roundabout production process
- μ measures the ease with which inputs can be substituted. This is the source of market power later on

Final goods sector (2)

- Pricing decision:

$$P_Y(t) \equiv N(t)^{-\eta} \cdot \left[N(t)^{\mu/(1-\mu)} \sum_{i=1}^{N(t)} P_i(t)^{1/(1-\mu)} \right]^{1-\mu}$$

- Derived demand for input i (for $(i = 1, 2, \cdots, N(t))$):

$$\frac{X_i(t)}{Y(t)} = N(t)^{(\eta-\mu)/(\mu-1)} \cdot \left(\frac{P_i(t)}{P_Y(t)} \right)^{\mu/(1-\mu)}$$

So $\mu/(1-\mu)$ is the demand elasticity

Intermediate goods sector

- *Intermediate goods sector* (many small monopolistically competitive firms): Each firm (patent holder) uses labour to produce its own slightly unique variety of the intermediate input
- Technology:

$$X_i(t) = Z_X \cdot L_i(t)$$

  constant marginal production costs
- Pricing decision:

$$P_i(t) = \mu \cdot \frac{W(t)}{Z_X}$$

  where μ is the gross monopoly markup

R & D sector

- *R&D sector* (CRTS, perfectly competitive, external effect "standing on the shoulder of giants"): Produces blueprints for new intermediate inputs, using labour as an input
- Technology:

$$\dot{N}(t) = Z_R \cdot N(t) \cdot L_R(t)$$

  Labour engaged in the R&D sector becomes more productive as more patents already exist. Today's engineers "stand on the shoulders of giants"
- Pricing decision:

$$P_N(t) = \frac{(1 - s_R) \cdot W(t)}{Z_R \cdot N(t)}$$

Household behaviour

- Representative infinitely-lived households
- Lifetime utility function:

$$\Lambda(0) = \int_0^{\infty} \frac{C(t)^{1-1/\sigma} - 1}{1 - 1/\sigma} \cdot e^{-\rho t} dt$$

- Household budget identity:

$$P_Y(t)C(t) + P_N(t)\dot{N}(t) = W(t)L_0 - T(t) + N(t)\bar{\Pi}(t)$$

- Optimality conditions:

$$\frac{\dot{C}(t)}{C(t)} = \sigma \cdot [r(t) - \rho]$$

$$r(t) = \frac{\bar{\Pi}(t) + \dot{P}_N(t)}{P_N(t)}$$

where $r(t)$ is the rate of return on blueprints

Loose ends

- Final goods market:

$$Y(t) = C(t) + G(t), \qquad G(t) = gY(t)$$

  where g is a policy variable
- Government budget constraint:

$$T(t) = G(t) + s_R W(t) L_R(t)$$

- Labour market equilibrium:

$$L_X(t) + L_R(t) = L_0$$

- But $Z_X L_X(t) = N(t)\bar{X}(t)$ and $Z_R L_R(t) = \dot{N}(t)/N(t)$ so:

$$\frac{\dot{N}(t)}{N(t)} = Z_R \left[L_0 - \frac{N(t)\bar{X}(t)}{Z_X} \right] > 0$$

  Assumption: intermediate sector does not absorb the entire labour force

Solving the model (1)

- *Step 1.* Intermediate results:

$$\frac{\bar{\Pi}(t)}{P_N(t)} = (\mu - 1)\frac{Z_R}{1 - s_R} L_X(t)$$

$$\frac{\dot{P}_N(t)}{P_N(t)} = (\eta - 2)\frac{\dot{N}(t)}{N(t)}$$

$$C(t) = (1 - g)\, N(t)^{\eta - 1} Z_X L_X(t)$$

- *Step 2.* Dynamic equations:

$$\gamma_C(t) = \sigma \left[(\mu - 1)\, \frac{Z_R}{1 - s_R} L_X(t) + (\eta - 2)\gamma_N(t) - \rho \right]$$

$$\gamma_C(t) = (\eta - 1)\gamma_N(t) + \frac{\dot{L}_X(t)}{L_X(t)}$$

$$\gamma_N(t) = Z_R \left[L_0 - L_X(t) \right]$$

Solving the model (2)

- *Step 3.* Combine dynamic equations to get:

$$\frac{\dot{L}_X(t)}{L_X(t)} = Z_R \cdot \left[\frac{\sigma(\mu-1)}{1-s_R} + \eta - 1 + \sigma(2-\eta) \right] (L_X(t) - L_X^*) \quad \text{(A)}$$

where L_X^* is defined as:

$$L_X^* = \frac{[\eta - 1 + \sigma(2-\eta)] L_0 + \sigma\rho/Z_R}{\sigma(\mu-1)/(1-s_R) + \eta - 1 + \sigma(2-\eta)}$$

- Eq. (A) is an unstable differential equation in $L_X(t)$. Only economically sensible solution is $L_X(t) = L_X^*$
- Hence $\gamma_N(t) = \gamma_N^* \equiv Z_R [L_0 - L_X^*]$ is also time-invariant. There is no transitional dynamics (no capital)

Economic growth

- The growth rates are:

$$\gamma_N^* = \frac{\dfrac{\mu - 1}{1 - s_R} Z_R L_0 - \rho}{\dfrac{\mu - 1}{1 - s_R} + \dfrac{\eta - 1}{\sigma} + (2 - \eta)} > 0$$

$$\gamma_C^* = \gamma_Y^* = (\eta - 1)\gamma_N^*$$

- The innovation rate, γ_N^*:
  - increases with the monopoly markup (μ) and the subsidy (s_R)
  - increases with the size of the labour force (L_0)
  - (provided $\eta > 1$) increases with the intertemporal substitution elasticity (σ)
  - decreases with the rate of time preference (ρ)
- Consumption and aggregate output grow only if the returns to specialization are operative (so that $\eta > 1$)

Efficiency (1)

- Not obvious that the decentralized market equilibrium is efficient as there are both external effects and non-competitive behaviour
- "Quick-and-dirty" intuition would seem to suggest that there is too little innovation (under-investment in R&D) because the innovator does not capture all the beneficial effects of his act
- Formal approach in economic theory:
    - Compute what kind of allocation a (benevolent) social planner would choose
    - Such a planner takes into account ("internalizes") all external effects/economies of scale
    - Compare socially optimal allocation with decentralized market allocation
    - How can social optimum be replicated in the market?

Efficiency (2): The social optimum

- Social planner imposes symmetry up front and works directly with aggregates
- Current-value Hamiltonian:

$$\mathcal{H}_C(t) = \frac{\left[N(t)^{\eta-1} Z_X L_X(t)\right]^{1-1/\sigma} - 1}{1 - 1/\sigma} + \mu_N(t) N(t) Z_R \left[L_0 - L_X(t)\right]$$

where $\mu_N(t)$ is the co-state variable for $N(t)$

- FONCs:
  - For $L_X(t)$:

$$Z_R \mu_N(t) = \frac{Z_X N(t)^{\eta-2}}{\left[N(t)^{\eta-1} Z_X L_X(t)\right]^{1/\sigma}}$$

  - For $N(t)$:

$$\dot{\mu}_N(t) = \rho \mu_N(t) - \frac{(\eta-1) Z_X L_X(t) N(t)^{\eta-2}}{\left[N(t)^{\eta-1} Z_X L_X(t)\right]^{1/\sigma}} - \mu_N(t) Z_R \left[L_0 - L_X(t)\right]$$

Efficiency (3): The social optimum

- Combine the FONCs:

$$\frac{\dot{L}_X(t)}{L_X(t)} = (\eta - 1)\, Z_R L_X(t) - (\eta - 1)(1 - \sigma) Z_R L_0 - \sigma\rho$$

- Provided $\eta > 0$ this is an unstable differential. Optimal solution is time invariant:

$$L_X^{SO} = (1 - \sigma) L_0 + \frac{\sigma\rho}{(\eta - 1)\, Z_R}$$

- Optimal innovation rate:

$$\gamma_N^{SO} \equiv Z_R \left[L_0 - L_X^{SO}\right] = \sigma Z_R L_0 - \frac{\sigma\rho}{\eta - 1} > 0$$

$$\gamma_C^{SO} = \gamma_Y^{SO} = (\eta - 1)\gamma_N^{SO}$$

Efficiency (4): The comparison

- (For simple case, $\sigma = 1$) we find:

$$\gamma_N^*(s_R) \equiv \frac{(\mu - 1) Z_R L_0 - \rho (1 - s_R)}{\mu - s_R}$$

$$\gamma_N^{SO} = \frac{(\eta - 1) Z_R L_0 - \rho}{\eta - 1}$$

- Point of view #1: Suppose $s_R = 0$

$$\mu \cdot \left[\gamma_N^{SO} - \gamma_N^*(0)\right] = Z_R L_0 - \rho \cdot \frac{\mu - (\eta - 1)}{\eta - 1}$$

  - no general conclusion. $\gamma_N^{SO} > \gamma_N^*(0)$ or $\gamma_N^{SO} < \gamma_N^*(0)$?
  - if $\eta = \mu$ (knife-edge case): Q&D intuition is OK! We find that $\gamma_N^{SO} > \gamma_N^*(0)$
  - if $\eta \approx 1$ (weak specialization effect): there may be too much innovation!

Efficiency (5)

- Point of view #2: For which subsidy value do we find $\gamma_N^{SO} = \gamma_N^* (s_P)$?

$$\frac{s_R^*}{1 - s_R^*} = \frac{(\eta - 1) Z_R L_0 - \rho [\mu - (\eta - 1)]}{\rho (\mu - 1)} = \frac{\mu (\eta - 1)}{\rho (\mu - 1)} \cdot [\gamma_N^{SO} - \gamma_N^* (0)]$$

  - it is optimal to subsidize (tax) R&D labour if the *laissez-faire* economy innovates too slowly (quickly) relative to the social optimum

Counterfactual prediction (1)

- *Problematic aspect*: The growth rate depends on the scale of the economy (L_0 in this case). Hence, large countries should grow faster than small countries. This is not observed in reality. Jones removes the scale effect by replacing the R&D technology by:

$$\dot{N}(t) = Z_R \cdot L_R(t) \cdot N(t)^{\phi_1} \cdot \left[\bar{L}_R(t)\right]^{\phi_2 - 1}$$

  where $\bar{L}_R$ is average R&D labour per R&D firm
- We had $\phi_1 = 1$ but now assume $0 < \phi_1 < 1$ (the giants don't grow forever)
- We had $\phi_2 = 1$ but now assume $0 < \phi_2 \leq 1$ (duplication externality: individual R&D firms think the production function is linear, but in actuality it features diminishing returns to labour)

Counterfactual prediction (2)

- Assuming that the population grows at an exponential rate, n_L, the growth rates are now (see book for details):

$$\gamma_N^* = \frac{\phi_2 n_L}{1-\phi_1}$$
$$\gamma_y^* = \gamma_Y^* - n_L = (\mu - 1)\gamma_N^*$$
$$\gamma_c^* = \gamma_C^* - n_L = \gamma_y^*$$

- We reach the striking conclusion that by eliminating the scale effect we are back in the realm of exogenous growth and the Solow model!

Punchlines (1)

- Physical "capital-fundamentalist" models
  - Endogenous growth is possible if the substitution elasticity between capital and labour is relatively large. Such a model features transitional dynamics but violates Kaldor's stylized facts SF3 and SF5
  - The "aggregate production function" may be linear in the aggregate private capital stock if (a) there are very strong interfirm productivity spillovers or (b) the government maintains the right amount of public infrastructure. Knife-edge models of this type may or may not display transitional dynamics

Punchlines (2)

- Human-plus-physical capital models
  - Workers can improve the quality of their labour by schooling themselves
  - Schooling is costly because it leads to foregone labour earnings ("learning **or** doing approach")
  - Public subsidization schooling activities is good for steady-state economic growth
  - Models of this type display complicated transitional dynamics (because there are two stocks rather than one)

Punchlines (3)

- R&D based model of endogenous economic growth
- Model components:
  - Imperfectly competitive firms: endogenous price setting
  - R&D sector produces new blueprints
  - External effects: "shoulders of giants", returns to specialization
- Results:
  - Innovation leads to wider range of inputs
  - More "roundabout" production process gives rise to aggregate productivity gains
  - Subsidizing R&D activities is good for growth
  - There may be excessive innovation
- Scale effect can be removed