Foundations of Modern Macroeconomics Third Edition

Chapter 13: Exogenous economic growth – Ramsey-Cass-Koopmans

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Outline

The Ramsey-Cass-Koopmans model

- *Key idea*: replace the *ad hoc* savings (consumption) function by forward-looking theory based on dynamic utility maximization
- Important contributors:
  - Frank Ramsey (1903-1930): (1927) "A mathematical theory of saving," *Economic Journal*, **37**, 47–61
  - Tjalling Koopmans (1910-1985): (1965) "On the concept of optimal economic growth." In: *The Econometric Approach to Development Planning*. Chicago: Rand-McNally
  - David Cass (1937-2008): (1965) "Optimum growth in an aggregative model of capital accumulation," *Review of Economic Studies*, **32**, 233–240
- Building blocks of the model:
  - Decisions of a "representative household"
  - Decisions of a "representative firm"
  - Competitive market equilibrium

Representative consumer (1)

- Household / consumer used interchangeably [unitary decision making]
- Infinitely lived
- Identical
- Perfect foresight
- "Felicity function" (instantaneous utility):

$$U(c(t))$$

... with properties:

  - Positive but diminishing marginal felicity of consumption:

$$U'(c(t)) > 0$$
$$U''(c(t)) < 0$$

Representative consumer (2)

- . . . with properties:
  - "Inada-style" curvature conditions:

$$\lim_{c(t)\to 0} U'(c(t)) = +\infty$$
$$\lim_{c(t)\to \infty} U'(c(t)) = 0$$

- Labour supply is exogenous and grows exponentially:

$$\frac{\dot{L}(t)}{L(t)} = n$$

(May also be interpreted as dynastic family which grows exponentially at an exogenously give rate n)

Representative consumer (3)

- Utility functional is discounted integral of present and future felicity:

$$\Lambda(0) \equiv \int_0^{\infty} U(c(t))e^{-\rho t}dt, \quad \rho > 0 \tag{S1}$$

where $t = 0$ is the planning period ("today") and ρ is the rate of pure time preference ("Millian" welfare function–utility of representative family member)

- The budget identity is:

$$C(t) + \dot{A}(t) = r(t)A(t) + w(t)L(t)$$

where $C(t) \equiv L(t)c(t)$ is aggregate consumption, $A(t)$ is financial assets, $r(t)$ is the rate of return on these assets, $w(t)$ is the wage rate, and $\dot{A}(t) \equiv dA(t)/dt$

  - LHS: Uses of income: consumption plus saving
  - RHS: Sources of income: interest income plus wage income

Representative consumer (4)

- In *per capita* form the budget identity becomes:

$$\dot{a}(t) \equiv [r(t) - n]\, a(t) + w(t) - c(t) \qquad \text{(S2)}$$

where $a(t) \equiv A(t)/L(t)$ is per capita financial assets

  - **Note**: equation (S2) is just an *identity*; it does not restrict anything (e.g. household could accumulate debt indefinitely and let $a(t)$ approach $-\infty$)

- The solvency condition is the true *restriction* faced by the household:

$$\lim_{t\to\infty} a(t) \exp\left[-\int_0^t [r(\tau) - n]\, d\tau\right] = 0 \qquad \text{(S3)}$$

Loosely put, the household does not plan to “expire” with positive assets and is not allowed by the capital market to die hopelessly indebted

Representative consumer (5)

- By integrating (S2) over the (infinite) life time of the agent and taking into account the solvency condition (S3), we obtain the household life-time budget constraint:

$$\underbrace{\int_0^\infty c(t)e^{-[R(t)-nt]}dt}_{(a)} = \underbrace{a(0)+h(0)}_{(b)}, \quad R(t) \equiv \int_0^t r(\tau)d\tau \quad \text{(HBC)}$$

where $a(0)$ is the initial level of financial assets, $R(t)$ is a discounting factor, and $h(0)$ is human wealth:

$$h(0) \equiv \underbrace{\int_0^\infty w(t)e^{-[R(t)-nt]}dt}_{(c)} \qquad \text{(HW)}$$

(a) Present value of the lifetime consumption path
(b) Total wealth in planning period (financial plus human wealth)
(c) Human wealth: present value of wage; market value of time endowment per capita

Representative consumer (6)

- The consumer chooses a time path for $c(t)$ (for $t \in [0, \infty)$) which maximizes life-time utility, $\Lambda(0)$, subject to the life-time budget restriction. The first-order conditions are the budget restriction and:

$$\underbrace{U'\left(c(t)\right) \cdot e^{-\rho t}}_{(a)} = \underbrace{\lambda \cdot e^{-[R(t)-nt]}}_{(b)} \tag{S4}$$

where λ is the marginal utility of wealth, i.e. the Lagrange multiplier associated with the life-time budget restriction

(a) Marginal contribution to life-time utility (evaluated from the perspective of "today," i.e. $t = 0$) of consumption in period t

(b) Life-time marginal utility cost of consuming $c(t)$ rather than saving it. (The marginal unit of $c(t)$ costs $e^{-[R(t)-nt]}$ from the perspective of today. This cost is translated into utility terms by multiplying it with the marginal utility of wealth.)

Representative consumer (7)

- Since λ is constant (i.e. it does not depend on t), differentiation of (S4) yields an expression for the optimal *time profile* of consumption:

$$\frac{d}{dt}U'\left(c(t)\right) = -\lambda e^{-[R(t)-nt-\rho t]}\left[\frac{dR(t)}{dt} - n - \rho\right] \quad \Leftrightarrow$$

$$U''\left(c(t)\right)\frac{dc(t)}{dt} = -U'\left(c(t)\right)\left[r(t) - n - \rho\right] \quad \Leftrightarrow$$

$$\theta(c(t)) \cdot \frac{1}{c(t)}\frac{dc(t)}{dt} = r(t) - n - \rho \qquad \text{(S5)}$$

  - We have used the fact that $dR(t)/dt = r(t)$
  - $\theta(\cdot)$ is the *elasticity of marginal utility* which is positive for all positive consumption levels [strict concavity of $U(\cdot)$]:

$$\theta(c(t)) \equiv -\frac{U''(c(t))c(t)}{U'(c(t))}$$

Representative consumer (8)

- The *intertemporal substitution elasticity*, $\sigma(\cdot)$, is the inverse of $\theta(\cdot)$. Hence, (S5) can be re-written to yield the consumption Euler equation:

$$\frac{1}{c(t)}\frac{dc(t)}{dt} = \sigma(c(t)) \cdot \Big[r(t) - n - \rho\Big]$$

Intuition:

  - if $\sigma(\cdot)$ is low, a large interest gap $(r(t) - n - \rho)$ is needed to induce the household to adopt an upward sloping time profile for consumption. In that case the willingness to substitute consumption across time is low, the elasticity of marginal utility is high, and the marginal utility function has a lot of curvature
  - The opposite holds if $\sigma(\cdot)$ is high. Then, the marginal utility function is almost linear so that a small interest gap can explain a large slope of the consumption profile

Digression: specific forms of the felicity function

- Macroeconomists use specific functional forms for the felicity function in order to get closed-form solutions and to facilitate the computations. Two such forms are used:
  - The *exponential felicity* function (which features $\sigma(\cdot) = \alpha/c(t)$):

$$U\left(c(t)\right) \equiv -\alpha e^{-(1/\alpha)c(t)}, \qquad \alpha > 0$$

  - The *iso-elastic* felicity function (which features $\sigma(\cdot) = \sigma$):

$$U\left(c(t)\right) \equiv \begin{cases} \dfrac{c(t)^{1-1/\sigma} - 1}{1 - 1/\sigma} & \text{for } \sigma > 0, \quad \sigma \neq 1 \\ \ln c(t) & \text{for } \sigma = 1 \end{cases}$$

Digression: specific forms of the fecility function

- The corresponding consumption Euler equations are:

$$\frac{dc(t)}{dt} = \alpha\left[r(t) - n - \rho\right] \qquad \text{(exponential felicity)}$$

$$\frac{1}{c(t)}\frac{dc(t)}{dt} = \sigma\left[r(t) - n - \rho\right] \qquad \text{(iso-elastic felicity)}$$

- Most of our discussion will make use of the iso-elastic felicity function

Representative firm (1)

- Perfect competition
- Constant returns to scale
- Representative firm
- No adjustment costs on investment
- The stock market value of the firm is given by the discounted value of its cash flows:

$$V(0) = \int_0^\infty \Big[F\left(K(t), L(t)\right) - w(t)L(t) - I(t)\Big] e^{-R(t)} dt$$

  where $R(t)$ is the discounting factor given above and $I(t)$ is gross investment:

$$I(t) = \delta K(t) + \dot{K}(t)$$

- The firm maximizes $V(0)$ subject to the capital accumulation constraint

Representative firm (2)

- Since there are no adjustment costs on investment the firm's decision about factor inputs is essentially a static one, i.e. the familiar marginal productivity conditions for labour and capital hold:

$$F_L\left(K(t), L(t)\right) = w(t)$$
$$F_K\left(K(t), L(t)\right) = r(t) + \delta$$

- By writing the production function in the intensive form, i.e. $f(k(t)) \equiv F\left(\frac{K(t)}{L(t)}, 1\right)$, we can rewrite the marginal products of capital and labour as follows:

$$F_K\left(K(t), L(t)\right) = f'(k(t))$$
$$F_L\left(K(t), L(t)\right) = f(k(t)) - k(t)f'(k(t))$$

Dynamic general equilibrium

- We now have the complete Ramsey model (see **Table 13.1**):

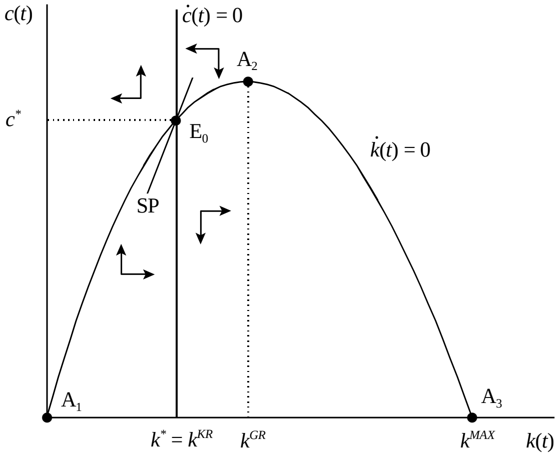
$$\frac{\dot{c}(t)}{c(t)} = \sigma \cdot [r(t) - n - \rho] \tag{T1.1}$$

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n)k(t) \tag{T1.2}$$

$$r(t) = f'(k(t)) - \delta \tag{T1.3}$$

  - Eqn (T1.1) is the consumption Euler equation
  - Eqn (T1.2) is the fundamental differential equation (FDE) for the capital stock
  - Eqn (T1.3) shows that the real interest rate is the net marginal product of capital
  - $c(t)$ is per capita consumption, $k(t)$ is the capital-labour ratio, and $r(t)$ is the interest rate
- **Figure 13.1** shows the phase diagram of the Ramsey model

Figure 13.1: Phase diagram of the Ramsey model



Capital dynamics

- Features of the $\dot{k} = 0$ line: equilibrium of capital stock per worker
  - Vertical in origin (Inada conditions)
  - Maximum in golden rule point A_2:

$$\left(\frac{dc(t)}{dk(t)}\right)_{\dot{k}(t)=0} = 0: \quad f'\left(k^{GR}\right) = \delta + n \qquad \text{(S6)}$$

  - Maximum attainable k is in point A_3 where k^{MAX} is:

$$\frac{f\left(k^{MAX}\right)}{k^{MAX}} = \delta + n$$

  - Capital dynamics:

$$\frac{\partial \dot{k}(t)}{\partial k(t)} = \underbrace{f' - (\delta}_{r} + n) = r - n \gtreqless 0 \text{ for } k(t) \lesseqgtr k^{GR}$$

See the horizontal arrows in Figure 13.1

Consumption dynamics

- Features of the $\dot{c} = 0$ line: flat per capita consumption profile
  - $\dot{c} = 0$ implies constant interest rate ($r = \rho + n$) and unique capital-labour ratio:

$$f'(k^{KR}) = \delta + n + \rho \tag{S7}$$

    (KR stands for "Keynes-Ramsey")
  - For points to the left (right) of the $\dot{c} = 0$ line k is too low (high) and r is too high (low). See the vertical arrows in Figure 13.1
- Comparing (S6) and (S7) shows that $f'(k^{KR}) > f'(k^{GR})$ so that $k^{KR} < k^{GR}$ and thus $r^{KR} > r^{GR}$ (no dynamic inefficiency possible!!)
- By removing the ad hoc savings function from the Solow-Swan model the possibility of oversaving vanishes

Economic growth properties

- The configuration of arrows in Figure 13.1 confirms that the Ramsey model is saddle point stable
  - $k(t)$ is the predetermined or "sticky" variable
  - $c(t)$ is the non-predetermined or "jumping" variable
- The equilibrium is unique (at point E_0)
- The saddle path is upward sloping
- In the BGP the model features a constant capital-labour ratio (k^{KR}). Hence, growth is just like in the Solow-Swan model: all variables grow at the same exogenously given rate (n in the present model as we assume $n_Z = 0$)

How fast is convergence? (1)

- How can we study convergence speed in a system of differential equations?
- We can study the convergence speed of the Ramsey model by linearizing it around the steady state:

$$\begin{bmatrix} \dot{c}(t) \\ \dot{k}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \sigma c^* f''(k^*) \\ -1 & \rho \end{bmatrix}}_{\Delta} \cdot \begin{bmatrix} c(t) - c^* \\ k(t) - k^* \end{bmatrix}$$

where Δ is the Jacobian matrix. Features:

- tr(Δ)$\equiv \lambda_1 + \lambda_2 = \rho > 0$ and $|\Delta| \equiv \lambda_1 \lambda_2 = \sigma c^* f''(k^*) < 0$, where λ_1 and λ_2 are the characteristic roots of Δ
- Hence, the model is saddle point stability, i.e. λ_1 and λ_2 have opposite signs

How fast is convergence? (2)

- . . . Features
  - The absolute value of the stable (negative) characteristic root determines the approximate convergence speed of the economic system. After some manipulation we obtain the following expression:

$$\beta \equiv \frac{\rho}{2}\left[\sqrt{1-\frac{4\sigma c^* f''(k^*)}{\rho^2}}-1\right]$$
$$=\frac{\rho}{2}\left[\sqrt{1+\frac{4}{\rho^2}\cdot\frac{\sigma}{\sigma_{KL}}\cdot\left(\frac{c}{k}\right)^*\cdot(r^*+\delta)\cdot(1-\omega_K)}-1\right]$$

  - By plugging in some realistic numbers for the structural parameters (σ, σ_{KL}, ρ, δ, etcetera) we can compute the speed of convergence implied by the Ramsey model. See **Table 13.2**. Only if the capital share is high and the intertemporal substitution elasticity is relatively small (low σ) does the model predict a realistic convergence speed

Table 13.2: Convergence speed in the RCK model

	σ/σ_{KL}			
	0.2	0.5	1	2
$\omega_K = \frac{1}{3}$	4.23	7.38	10.97	16.08
$\omega_K = \frac{1}{2}$	2.41	4.39	6.70	10.00
$\omega_K = \frac{2}{3}$	1.25	2.44	3.88	5.96

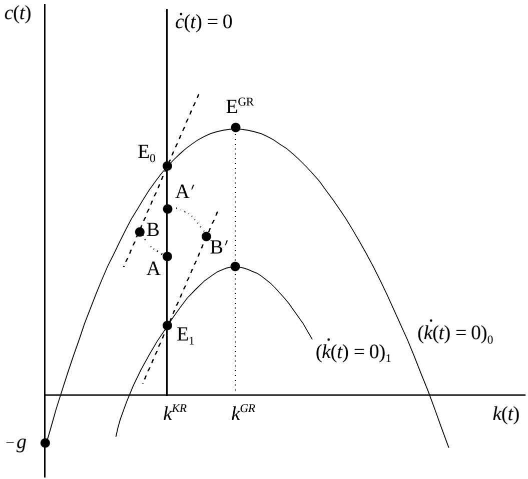
Fiscal policy in the RCK model (1)

- Let us introduce the government
- Suppose the government consumes goods, $g(t)$. The FDE for $k(t)$ becomes:

$$\dot{k}(t) = f(k(t)) - c(t) - g(t) - (\delta + n)k(t)$$

  where $g(t) \equiv G(t)/L(t)$ is per capita government consumption
- Government consumption withdraws resources which are no longer available for private consumption or replacement of the capital stock. As a result, for a given level of per capita public consumption, $g(t) = g$, the $\dot{k}(t) = 0$ line can be drawn as in **Figure 13.2**. Features:
  - Still no dynamic inefficiency
  - Still unique equilibrium

Figure 13.2: Fiscal policy in the RCK model



Fiscal policy in the RCK model (2)

- Heuristic solution concept can be used
- An *unanticipated* and *permanent* increase in the level of government consumption per worker shifts the $\dot{k}(t) = 0$ line down, say to $(\dot{k}(t) = 0)_1$
  - Since the shock comes as a complete surprise to the representative household, it reacts to the increased level of taxes (needed to finance the additional government consumption) by cutting back private consumption. The representative household feels poorer as a result of the shock and, as consumption is a normal good, reduces it one-for-one:

$$\frac{dc(t)}{dg} = -1, \quad \frac{dy(t)}{dg} = \frac{dk(t)}{dg} = 0, \qquad \forall t \in [0, \infty)$$

  - There is no transitional dynamics because the shock itself has no long-run effect on the capital stock and there are no anticipation effects. In terms of Figure 13.2 the economy jumps from E_0 to E_1

Fiscal policy in the RCK model (3)

- A *temporary* increase in g causes non-trivial transition effects
  - The representative household anticipates the temporarily higher taxes but spreads the negative effect on human wealth out over the entire life-time consumption path. The impact effect on private consumption is still negative but less than one-for-one:

$$-1 < \frac{dc(0)}{dg} < 0 \quad \text{(jump from E}_0\text{ to A)}$$

  - Immediately after the shock the household starts to dissave so that the capital stock falls, the interest rate rises, and (by (T1.1)) consumption rises over time. The economy moves from A to B which is reached at the time the government consumption is cut back to its initial level again
  - This cut in g (and the associated taxes) releases resources which allows the capital stock to return to its constant steady-state level (from B to E_0)
  - Bottom line: there is a temporary decline in output per worker

Fiscal policy in the RCK model (4)

- With an *anticipated* and *permanent* increase in g the opposite effect occurs during transition
  - Consumption falls by less than one-for-one, but since the government consumption has not risen yet it leads to additional saving and a gradual increase in the capital stock, a reduction in the interest rate, and a downward-sloping consumption profile
  - At impact the economy jumps from E_0 to A', after which it gradually moves from A' to B' during transition
  - Point B' is reached at precisely the time the policy is enacted. As g is increased net saving turns into net dissaving and the capital stock starts to fall. The economy moves from point B' to E_1
  - Hence, there is a temporary boost to k due to anticipation effects

Ricardian equivalence in the RCK model (1)

- Ricardian equivalence holds in the RCK model
- The government budget *identity* (in per capita form) is given by:

$$\dot{b}(t) = [r(t) - n]\, b(t) + g(t) - \tau(t) \qquad \text{(S8)}$$

- Like the representative household, the government must also remain solvent so that it faces an intertemporal solvency condition of the following form:

$$\lim_{t\to\infty} b(t) e^{-[R(t)-nt]} = 0 \qquad \text{(S9)}$$

By combining (S8) and (S9), we obtain the government budget *restriction*:

$$b(0) = \int_0^\infty [\tau(t) - g(t)]\, e^{-[R(t)-nt]} dt \qquad \text{(GBC)}$$

Ricardian equivalence in the RCK model (2)

- To the extent that there is a pre-existing government debt $(b(0) > 0)$, solvency requires that this debt must be equal to the present value of future primary surpluses. In principle, there are infinitely many paths for $\tau(t)$ and $g(t)$ (and hence for the primary deficit), for which the GBC is satisfied
- The budget *identity* of the representative agent is:

$$\dot{a}(t) \equiv [r(t) - n]\, a(t) + w(t) - \tau(t) - c(t)$$

- The household solvency condition is:

$$\lim_{t\to\infty} a(t) \exp\left[-\int_0^t [r(\tau) - n]\, d\tau\right] = 0$$

Ricardian equivalence in the RCK model (3)

- The household budget *restriction* is then:

$$\int_0^\infty c(t)e^{-[R(t)-nt]}dt = a(0) + h(0) \qquad \text{(HBC)}$$

$$h(0) \equiv \int_0^\infty [w(t) - \tau(t)]\, e^{-[R(t)-nt]}dt$$

- By using the GBC, human wealth can be rewritten as:

$$h(0) = \int_0^\infty [w(t) - g(t)]\, e^{-[R(t)-nt]}dt - b(0) \qquad \text{(S10)}$$

- The path of lump-sum taxes completely vanishes from the expression for human wealth!
- Since $b(0)$ and the path for $g(t)$ are given, the particular path for lump-sum taxes does not affect the total amount of resources available to the representative agent. As a result, the agent's real consumption plans are not affected either

Ricardian equivalence in the RCK model (4)

- By using (S10) in the HBC, the household budget restriction can be written as:

$$\int_0^\infty c(t)e^{-[R(t)-nt]}dt = [a(0)-b(0)] + \int_0^\infty [w(t)-g(t)]\cdot e^{-[R(t)-nt]}dt$$

- Under Ricardian equivalence, government debt should not be seen as household wealth, i.e. $b(0)$ must be deducted from total financial wealth in order to reveal the household's true financial asset position

- The RCK model can easily be extended to the open economy. Two complications in a small open economy (SOE) setting:
  - World interest rate, $\bar{r}$, constant, so physical capital stock jumps across borders unless we are willing to assume adjustment costs on investment
  - Consumption steady state only defined if $\bar{r}$ is equal to $\rho + n$ (knife-edge case). If $\bar{r} > \rho + n$ (patient country) then country ends up owning all assets; if $\bar{r} < \rho + n$ (impatient country) then country slowly disappears
  - If $\bar{r} = \rho + n$ then there is a zero root in consumption (as $\dot{c}(t)/c(t) = 0$ is the Euler equation): flat consumption profile. National solvency condition determines level of consumption. There is *hysteresis* in consumption and the stock of net foreign assets
- Section 13.4 gives an example: Investment stimulation in a small open economy. What are the key properties of this type of model?

RCK model for the small open economy (1)

- Knife-edge condition:

$$\bar{r} = \rho + n$$

- Representative firm faces adjustment costs of investment. Concave installation function:

$$\dot{K}(t) = \left[\Phi\left(\frac{I(t)}{K(t)}\right) - \delta \right] K(t)$$

with $\Phi(0) = 0$, $\Phi'(\cdot) > 0$, and $\Phi''(\cdot) < 0$

- Objective function of the firm:

$$V(0) = \int_0^{\infty} \Big[F\left(K(t), L(t)\right) - w(t)L(t) - \left(1 - s_I\right) I(t) \Big] e^{-\bar{r}t} dt$$

RCK model for the small open economy (2)

- Current-value Hamiltonian:

$$\mathcal{H}_C \equiv F\left(K(t), L(t)\right) - w(t)L(t) - \left(1 - s_I\right) I(t) + q\left(t\right) \cdot \left[\Phi\left(\frac{I(t)}{K(t)}\right) - \delta\right] \cdot K(t)$$

  - Control variables: $L\left(t\right)$, $I\left(t\right)$
  - State variable: $K\left(t\right)$
  - Co-state variable: $q\left(t\right)$

- FONCs:

$$w(t) = F_L\left(K(t), L(t)\right)$$

$$q(t)\Phi'\left(\frac{I(t)}{K(t)}\right) = 1 - s_I$$

$$\dot{q}(t) = \left[\bar{r} + \delta - \Phi\left(\frac{I(t)}{K(t)}\right)\right] q(t) - F_K\left(K(t), L(t)\right) + \left(1 - s_I\right) \frac{I(t)}{K(t)}$$

$$\lim_{t\to\infty} e^{-\bar{r}t} q\left(t\right) \geq 0, \qquad \lim_{t\to\infty} e^{-\bar{r}t} q\left(t\right) K\left(t\right) = 0$$

RCK model for the small open economy (3)

- Macroeconomic closure:
  - GDP in an open economy (no government consumption):

    $$Y(t) \equiv C(t) + I(t) + X(t)$$

    where $X(t)$ is net exports
  - GNP equals GDP plus interest earnings on net foreign assets, $\bar{r}A_F(t)$. The current account is:

    $$\dot{A}_F(t) = \bar{r}A_F(t) + X(t) = \bar{r}A_F(t) + Y(t) - C(t) - I(t)$$

  - In per capita terms ($a_F(t) \equiv A_F(t)/L(t)$ etcetera):

    $$\dot{a}_F(t) = \rho a_F(t) + y(t) - c(t) - i(t)$$

    where we have used $\bar{r} \equiv \rho + n$
  - Intertemporal solvency condition:

    $$\lim_{t \to \infty} a_F(t) e^{-\rho t} = 0$$

  - Model is given in **Table 13.3**

Table 13.3: The RCK model for the open economy

$$\frac{\dot{c}(t)}{c(t)} = 0 \tag{T3.1}$$

$$q(t) \cdot \Phi'\left(\frac{i(t)}{k(t)}\right) = 1 - s_I \tag{T3.2}$$

$$\begin{aligned} \dot{q}(t) &= \left[\rho + n + \delta - \Phi\left(\frac{i(t)}{k(t)}\right)\right] q(t) - f'(k(t)) \\ &\quad + \frac{(1 - s_I)\, i(t)}{k(t)} \end{aligned} \tag{T3.3}$$

$$\frac{\dot{k}(t)}{k(t)} = \Phi\left(\frac{i(t)}{k(t)}\right) - n - \delta \tag{T3.4}$$

$$\dot{a}_F(t) = \rho a_F(t) + f(k(t)) - c(t) - i(t) \tag{T3.5}$$

RCK model for the small open economy (4)

- System features a zero-root in the consumption Euler equation
- Solution method:
  - Consumers will always choose:

$$c\,(t) = c\,(0) \qquad \forall t \geq 0$$

  - Nation faces an intertemporal “budget constraint” of the form:

$$a_{F0} = \int_0^{\infty} \left[c(t) + i(t) - f\,(k(t))\right] e^{-\rho t} dt$$

  - Eqs. (T3.2)–(T3.4) can be solved independently from $c\,(t)$ and $a_F\,(t)$
  - By substituting the solutions for $i\,(t)$ and $k\,(t)$ into the budget constraint we can solve for $c\,(0)$

RCK model for the small open economy (5)

- What determines the adjustment speed in a SOE?
  - Convenient installation function:

$$\Phi\left(\frac{i(t)}{k(t)}\right) \equiv \frac{1}{1-\sigma_I}\left(\frac{i(t)}{k(t)}\right)^{1-\sigma_I}, \qquad 0 < \sigma_I < 1$$

    The lower is σ_I, the closer $\Phi(\cdot)$ resembles a straight line, and the less severe are adjustment costs
  - Implies investment demand:

$$\frac{i(t)}{k(t)} = g(q(t), s_I) \equiv \left(\frac{q(t)}{1-s_I}\right)^{1/\sigma_I}$$

  - Investment rises if $q(t)$ or s_I increases

RCK model for the small open economy (6)

- What determines the adjustment speed in a SOE?
  - Linearized (q, K)-dynamics:

$$\begin{bmatrix} \dot{k}(t) \\ \dot{q}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & i^*(1-s_I)/\left[(q^*)^2\sigma_I\right] \\ -f''(k^*) & \rho \end{bmatrix}}_{\Delta_I} \begin{bmatrix} k(t)-k^* \\ q(t)-q^* \end{bmatrix}$$

  - $\text{tr}\,(\Delta_I) = \rho > 0$ and $|\Delta_I| < 0$ so the system is saddle-point stable. See **Figure 13.3**
  - Stable (negative root) is finite due to the existence of adjustment costs which ensures that physical capital is immobile in the short run

Figure 13.3: Investment in the open economy

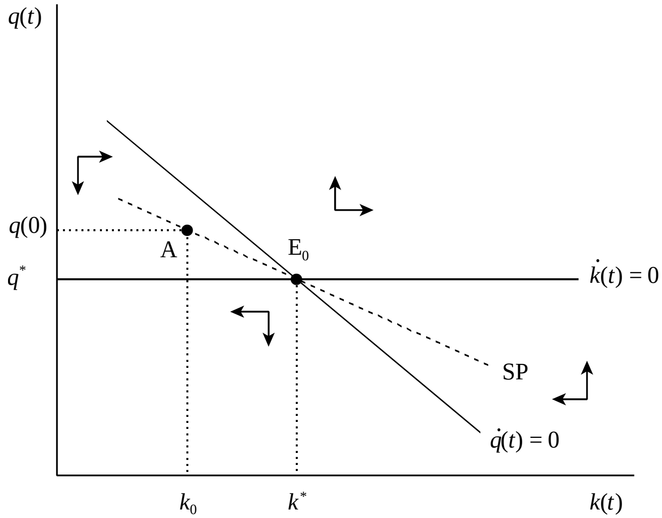
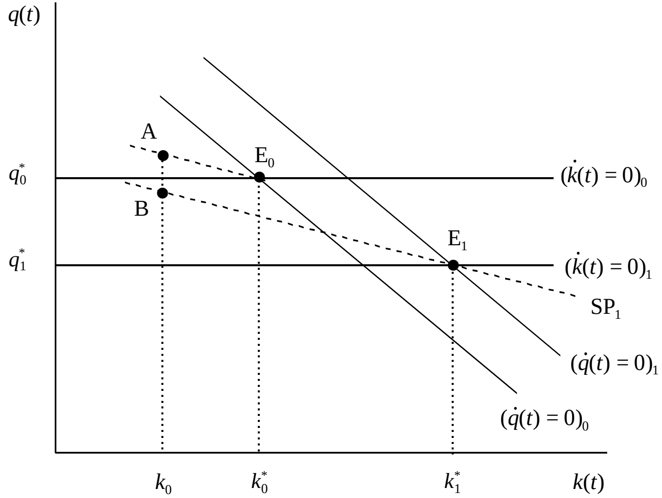


Figure 13.4: An investment subsidy with high mobility of physical capital



Household behaviour (1)

- We now turn to **the** most important workhorse model of modern macroeconomics: the Extended Ramsey-Cass-Koopmans model
- *Key idea*: To get an interesting theory of short-term fluctuations in macroeconomic variables such as output, employment, consumption, investment, wages, and the interest rate we must augment the RCK model by adding an endogenous labour supply decision by households
- Fortunately, endogenizing labour supply is straightforward. In what follows we abstract from population growth ($n = 0$ throughout)

Household behaviour (2)

- The household lifetime utility function is:

$$\Lambda(0) \equiv \int_0^\infty \Big[\varepsilon \ln C(t) + (1-\varepsilon) \ln [1 - L(t)]\Big] e^{-\rho t} dt \quad \text{(S11)}$$

  - The felicity function is logarithmic; sub-felicity is Cobb-Douglas: $C^\varepsilon [1-L]^{1-\varepsilon}$
  - **Note**: If we set $\varepsilon = 1$ we have the case with exogenous labour supply (the traditional RCK formulation)

- The agent's budget *identity* is:

$$\dot{A}(t) \equiv r(t)A(t) + w(t)L(t) - T(t) - C(t) \quad \text{(S12)}$$

where the only thing that has changed is that labour income is now wL (recall that L is a choice variable)

Household behaviour (3)

- Integrating (S12) yields the agent's budget restriction:

$$A(0) = \int_0^\infty [C(t) - w(t)L(t) + T(t)]\, e^{-R(t)} dt \qquad \text{(S13)}$$

where we have used $R(t) \equiv \int_0^t r(s)ds$ and:

$$\lim_{t\to\infty} A(t)e^{-R(t} = 0 \qquad \text{(NPG)}$$

  - Equation (S13) says that the present value of household "primary deficits" $(C - wL + T)$ equals the value of initial financial wealth in the planning period
  - NPG stands for "no Ponzi game". (Ponzi was a famous swindler who ran chain letter schemes and ended his life in jail.) This is the solvency condition
  - $R(t)$ is the discounting factor. Note that the real interest rate, r, is endogenous (depends on accumulation and on labour supply decisions by aggregate household sector)

Household behaviour (4)

- The household chooses paths for consumption, labour supply, and financial assets, such that lifetime utility is maximized subject to the household budget constraint (S13). In the text we derive the first-order conditions:

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho \tag{S14}$$

$$\frac{C(t)}{1 - L(t)} \frac{1 - \varepsilon}{\varepsilon} = w(t) \tag{S15}$$

  - Eq. (S14): dynamic part. Consumption Euler equation: the optimal time profile of consumption depends on the gap between the interest rate and the rate of time preference. If $r > \rho$, postpone consumption until later and adopt an upward sloping time profile of consumption
  - Eq. (S15): static part. The MRS between consumption and leisure should be equated to the real wage

The unit-elastic extended RCK model (1)

- The equations of the extended RCK model are given in **Table 13.4** Show table
- The extended RCK model can be analyzed with the aid of its phase diagram in **Figure 13.5** Show figure
- The diagram reveals that there is a unique equilibrium (at E_0) which is saddle-point stable. The saddle path (SP_0) is upward sloping. How do we know this?
- The $\dot{K}=0$ line represents (C,K) combinations for which *net* investment is zero
  - The golden rule capital stock is K^{GR} (maximum consumption point)
  - For points above (below) the $\dot{K}=0$ line, consumption is too high (too low), and net investment is negative (positive)–see the horizontal arrows

The unit-elastic extended RCK model (2)

- The $\dot{C} = 0$ line represents (C, K) combinations for which the consumption profile is flat, i.e. for which $r = \rho$
  - Recall that:

$$\begin{aligned} r &= F_K(K, L) - \delta \\ &= F_K\left(\frac{K}{L}, 1\right) - \delta \end{aligned}$$

    so $r = \rho$ implies a constant K/L ratio. Hence, w and Y/K are constant also along the $\dot{C} = 0$ line. This means that $C/[1 - L]$ is also constant. Combining all these "great ratios" yields the conclusion that the $\dot{C} = 0$ line is linear and downward sloping
  - For points above (below) the $\dot{C} = 0$ line, consumption is too high (too low), labour supply is too low (too high), and the K/L ratio is too high (too low), i.e. $r < \rho$ ($r > \rho$) and $\dot{C} < 0$ ($\dot{C} > 0$). See the vertical arrows

Figure 13.5: Phase diagram of the unit-elastic model

Go back

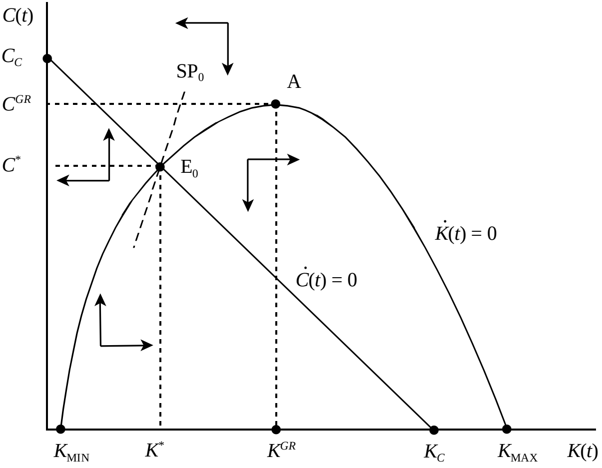


Table 13.4: The unit-elastic RCK model

Go back

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho \tag{T4.1}$$

$$\dot{K}(t) = Y(t) - C(t) - G(t) - \delta K(t) \tag{T4.2}$$

$$G(t) = T(t) \tag{T4.3}$$

$$w(t) = (1-\alpha)\frac{Y(t)}{L(t)} \tag{T4.4}$$

$$r(t) + \delta = \alpha\frac{Y(t)}{K(t)} \tag{T4.5}$$

$$L(t) = 1 - \frac{1-\varepsilon}{\varepsilon}\frac{C(t)}{w(t)} \tag{T4.6}$$

$$Y(t) = Z_0 K(t)^{\alpha} L(t)^{1-\alpha} \tag{T4.7}$$

Permanent increase in government consumption (1)

- To get to know the extended RCK model we study a simple example of fiscal policy: a permanent increase in government consumption financed by means of a lump-sum tax ($G \uparrow$ and $T \uparrow$)
- The *long-run effects* are obtained with back-of-the-envelope calculations
- $\dot{K} = 0$ implies $I^*/K^* = \delta$ (a constant)
- $\dot{C} = 0$ implies $r^* = \rho$, $Y^*/K^* = (\rho + \delta)/\alpha$ (constants)
- w^* and $C^*/[1 - L^*]$ constant (see above)

Permanent increase in government consumption (2)

- In summary:

$$\frac{dY(\infty)}{Y^*} = \frac{dK(\infty)}{K^*} = \frac{dI(\infty)}{I^*} = \frac{dL(\infty)}{L^*} = -\omega_{LL}^* \frac{dC(\infty)}{C^*} \qquad \text{(S16)}$$

where $\omega_{LL}^* \equiv [1-L^*]/L^*$ represents the intertemporal substitution elasticity of labour supply

- Using the GME locus, $Y(t) = C(t) + I(t) + G_0$ yields:

$$\frac{dY(\infty)}{Y^*} = \omega_C^* \frac{dC(\infty)}{C^*} + \omega_I^* \frac{dI(\infty)}{I^*} + \omega_G^* \frac{dG}{G_0} \qquad \text{(S17)}$$

where $\omega_C^* \equiv C^*/Y^*$, $\omega_I^* \equiv I^*/Y^*$, and $\omega_G^* \equiv G_0/Y^*$ $[\omega_C^* + \omega_I^* + \omega_G^* = 1]$

Permanent increase in government consumption (3)

- Combining (S16) and (S17) yields the multiplier for output, consumption, and the capital stock:

$$\frac{dY(\infty)}{dG} = \frac{1}{1-\omega_I^*+\omega_C^*/\omega_{LL}^*} > 0$$

$$-1 < \frac{dC(\infty)}{dG} = -\frac{\omega_C^*/\omega_{LL}^*}{1-\omega_I^*+\omega_C^*/\omega_{LL}^*} < 0$$

$$\frac{dK(\infty)}{dG} = \frac{1}{\delta}\frac{dI(\infty)}{dG} = \frac{\omega_I^*/\delta}{1-\omega_I^*+\omega_C^*/\omega_{LL}^*} > 0$$

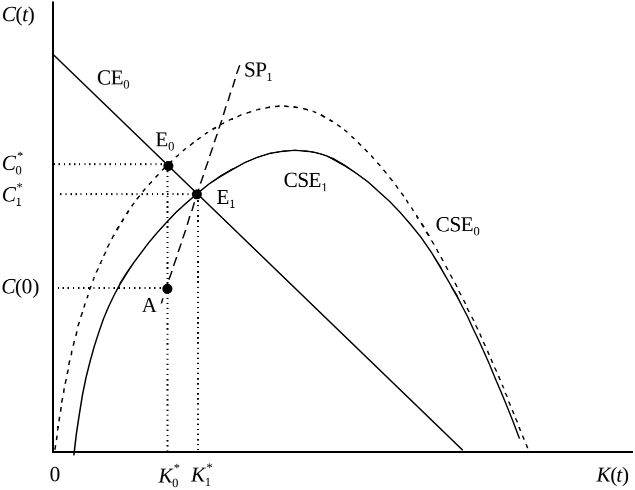
Permanent increase in government consumption (4)

- In the long run output is *crowded in* by additional government consumption, more so the more elastic is labour supply (the higher is ω_{LL}^{*})! Consumption is crowded out [though by less than one for one], and the capital stock rises. *Economic intuition:*
  - The household feels poorer because the lump-sum tax goes up permanently ($T \uparrow$)
  - This means that the value of human wealth falls ($H(\infty) \equiv [w - T]/r$), so that on that account consumption and leisure fall ($C \downarrow$ and $[1 - L] \downarrow$ and thus $L \uparrow$)
  - The drop in C makes it possible for the household to save more so that $K \uparrow$ (to restore constant K/L ratio)
  - Note the sharp contrast with the case of exogenous labour supply!

Permanent increase in government consumption (5)

- The *short-run effects* of the fiscal shock are more difficult to compute. We can use **Figure 13.6**
  - The capital stock equilibrium (CSE) line (for which $\dot{K} = 0$) shifts down because $G \uparrow$. Nothing happens to the consumption equilibrium (CE) line (for which $\dot{C} = 0$) because lump-sum taxes are used
  - At impact the capital stock is predetermined [at K_0^*] but the economy jumps to the new saddle path [from E_0 to A]. Households immediately adjust their consumption downward and their labour supply upwards due to the higher taxes
  - Since $C \downarrow$ and $L \uparrow$ (and thus $Y \uparrow$):
    - Households can save more [accumulate capital]: point A lies below CSE_1 so that $\dot{K} > 0$
    - The K/L ratio falls (capital relatively scarce) so that $r \uparrow$ and thus $\dot{C} > 0$ in point A
- *During transition* both C and K rise until the new equilibrium in E_1 is reached

Figure 13.6: Effects of fiscal policy



Quantification of the results (1)

- *Key idea*: Graphical methods are of limited use because they only yield *qualitative results* ("plus" or "minus"). We would like to know more, namely how large are the effects? We want *quantitative results*
- In the text we show in detail how we can obtain quantitative results (for impact, transitional, and long-run effects) by *linearizing the model*
- Advantages of linearizing:
  - We can solve the model for all kinds of shocks (not just fiscal)
  - The linearized model is expressed in terms of parameters which can be measured by econometric means (adding empirical content to the model), e.g. income shares of various macro variables (ω_C^*, ω_I^*, ω_G^*, ε_L) etcetera
  - We can simulate realistically calibrated models on the computer and see how well they fit the real world data (the *Lucas program*)

Details of the linearization method (1)

- Recall the formula of the first-order Taylor approximation of function $f(x)$ around a point x_0:

$$f(x) \approx f(x_0) + f'(x_0)\,[x - x_0]$$

- For a function with two variables we have:

$$f(x, y) \approx f(x_0, y_0) + f'_x(x_0, y_0)\,[x - x_0] + f'_y(x_0, y_0)\,[y - y_0]$$

  where $f_x(\cdot) \equiv \partial f(x, y)/\partial x$ and $f_y(\cdot) \equiv \partial f(x, y)/\partial y$ are the partial derivatives
- For obvious reasons linear expressions in Table 13.4 need no linearization
- Non-linear expressions are linearized around the steady-state values of variables appearing in it
- Using these results we get **Table S.1**

Table S1: The linearized Extended RCK model

$$\dot{C}(t) = C^* \left[r(t) - \rho\right] \tag{ST1.1}$$

$$\dot{K}(t) = Y^* \left[\frac{Y(t)-Y^*}{Y^*} - \omega_C^* \frac{C(t)-C^*}{C^*} - \omega_G^* \frac{G(t)-G_0}{G_0} - \omega_I^* \frac{K(t)-K^*}{K^*} \right] \tag{ST1.2}$$

$$\frac{w(t)-w^*}{w^*} = \frac{Y(t)-Y^*}{Y^*} - \frac{L(t)-L^*}{L^*} \tag{ST1.4}$$

$$r(t)-\rho = (\rho+\delta) \left[\frac{Y(t)-Y^*}{Y^*} - \frac{K(t)-K^*}{K^*} \right] \tag{ST1.5}$$

$$\frac{L(t)-L^*}{L^*} = \omega_{LL}^* \left[\frac{w(t)-w^*}{w^*} - \frac{C(t)-C^*}{C^*} \right] \tag{ST1.6}$$

$$\frac{Y(t)-Y^*}{Y^*} = \alpha \frac{K(t)-K^*}{K^*} + (1-\alpha) \frac{L(t)-L^*}{L^*} \tag{ST1.7}$$

Details of the linearization method (2)

- Using (ST1.4)–(ST1.7) we can find the Quasi-Reduced Form (QRF) expressions for the perturbations in employment, output, wages, and the interest rate in terms of the state variables (consumption and the capital stock)

$$\frac{L(t)-L^*}{L^*} = \frac{\omega_{LL}^*}{1+\alpha\omega_{LL}^*}\left[\alpha\frac{K(t)-K^*}{K^*} - \frac{C(t)-C^*}{C^*}\right]$$

$$\frac{Y(t)-Y^*}{Y^*} = \frac{1}{1+\alpha\omega_{LL}^*}\left[\alpha(1+\omega_{LL}^*)\frac{K(t)-K^*}{K^*} - (1-\alpha)\omega_{LL}^*\frac{C(t)-C^*}{C^*}\right]$$

$$\frac{w(t)-w^*}{w^*} = \frac{\alpha}{1+\alpha\omega_{LL}^*}\left[\frac{K(t)-K^*}{K^*} + \omega_{LL}^*\frac{C(t)-C^*}{C^*}\right]$$

$$r(t)-\rho = -\frac{(\rho+\delta)(1-\alpha)}{1+\alpha\omega_{LL}^*}\left[\frac{K(t)-K^*}{K^*} + \omega_{LL}^*\frac{C(t)-C^*}{C^*}\right]$$

Details of the linearization method (3)

- By using the QRF for $r(t)$ in (ST1) and the QRF for $Y(t)$ in (ST1.2) we obtain the fundamental system of linear differential equations for $C(t)$ and $K(t)$:

$$\begin{bmatrix} \dot{C}(t) \\ \dot{K}(t) \end{bmatrix} = \Delta \begin{bmatrix} C(t) - C^* \\ K(t) - K^* \end{bmatrix} - \begin{bmatrix} 0 \\ dG \end{bmatrix}$$

where we assume that government consumption features a stepwise increase equal to dG (i.e. $G_1 = G_0 + dG$

- The Jacobian matrix is:

$$\Delta \equiv \begin{bmatrix} -\dfrac{(1-\alpha)\omega_{LL}^*}{1+\alpha\omega_{LL}^*}(\rho+\delta) & -\dfrac{1-\alpha}{1+\alpha\omega_{LL}^*}(\rho+\delta)\dfrac{C^*}{K^*} \\ -\left(\dfrac{C^*}{Y^*} + \dfrac{(1-\alpha)\omega_{LL}^*}{1+\alpha\omega_{LL}^*}\right)\dfrac{Y^*}{C^*} & \left(\dfrac{C^*+G_0}{Y^*} - \dfrac{1-\alpha}{1+\alpha\omega_{LL}^*}\right)\dfrac{Y^*}{K^*} \end{bmatrix}$$

Quantification of the results (2)

- In the text we show what a reasonable calibration looks like and derive multipliers and elasticities. See **Table 13.5**.
- In the text we show how the loglinearized model can be used to study more difficult types of shocks. The worked example deals with *temporary fiscal policy* and its effects of the key macroeconomic variables. The degree of persistence of the shock critically influences the adjustment path. (See **Figures 13.8** for details of the derivation ▸ Show figures)
  - Consumption falls regardless of the degree of shock persistence
  - Capital rises initially if labour supply is highly elastic and the shock is relatively persistent
  - Capital falls initially if labour supply is not very elastic and the shock is relatively transitory

Table 13.5: Government consumption multipliers

Variable	*Impact effect*	*Long-run effect*
$\frac{dY}{dG}$	1.029	1.054
$\frac{dC}{dG}$	−0.539	−0.158
$\frac{dI}{dG}$	0.568	0.212
$\frac{dK}{K^*}/\frac{dG}{G}$	0	0.211
$\frac{dL}{L^*}/\frac{dG}{G}$	0.309	0.211
$\frac{dr}{r^*}/\frac{dG}{G}$	0.518	0
$\frac{dw}{w^*}/\frac{dG}{G}$	−0.103	0

Figure 13.7: Phase diagram of the linearized model

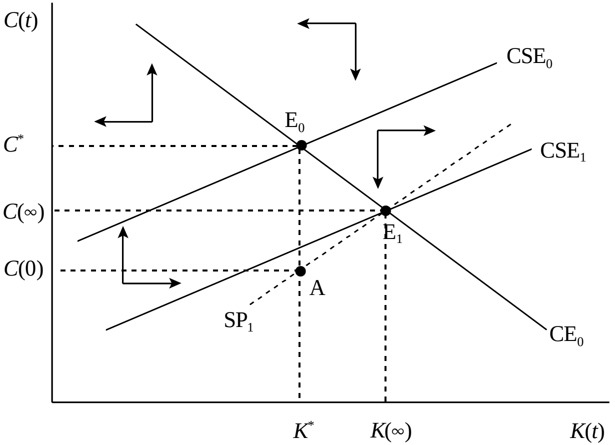
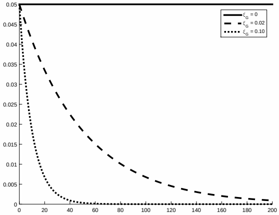
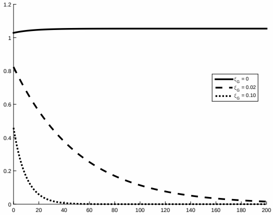


Figure 13.8: Temporary fiscal policy

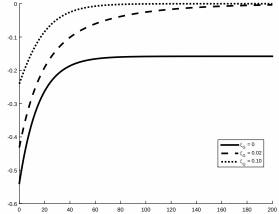
(a) Shock: $G(t) - G_0$



(b) Output: $\frac{Y(t)-Y^*}{dG}$



(c) Consumption: $\frac{C(t)-C^*}{dG}$



(d) Investment: $\frac{I(t)-I^*}{dG}$

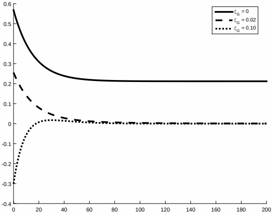
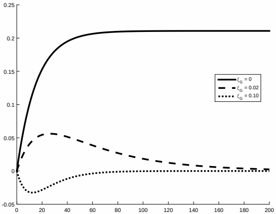


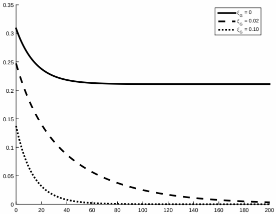
Figure 13.8: Temporary fiscal policy (continued)

Go back

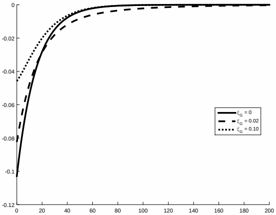
(e) Capital: $\frac{G_0}{K^*}\frac{K(t)-K^*}{dG}$



(f) Employment: $\frac{G_0}{L^*}\frac{L(t)-L^*}{dG}$



(g) Wage: $\frac{G_0}{w^*}\frac{w(t)-w^*}{dG}$



(h) Interest rate: $\frac{G_0}{r^*}\frac{r(t)-r^*}{dG}$

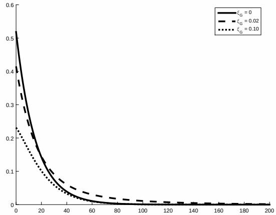
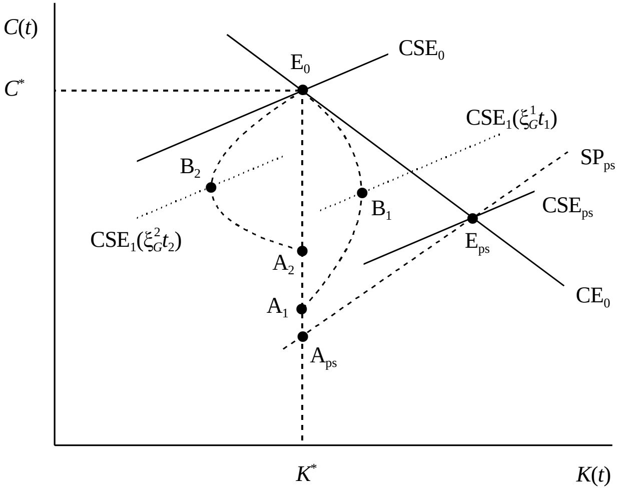


Figure 13.10: Phase diagram for temporary shock



Conclusions regarding the RCK model

- It yields very similar growth predictions as the Solow-Swan model does
- Unlike the Solow-Swan model it features Ricardian equivalence and rules out oversaving
- It is attractive because it features intertemporal optimization by households rather than ad hoc consumption-saving rules
- It forms the basis of much of modern macroeconomics (endogenous growth theory, RBC theory)
- It can easily be extended to the open economy. Two complications in a small open economy (SOE) setting
- But .... empirically the RCK consumption theory does not work very well (and the ad hoc rules do work well!)